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THE

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ON THE RELATION BETWEEN THE ANGULAR VELOCITY ABOUT AND THE ANGULAR VELOCITY OF THE INSTANTANEOUS AXIS OF A BODY REVOLVING SPONTANEOUSLY ABOUT A FIXED POINT, AND ON THE AXES OF GREATEST AND LEAST MOBILITY.

By WILLIAM WALTON, M.A., Fellow of Trinity Hall.

LET λ, μ, ν, be the direction-cosines of the instantaneous axis at the end of any time t in reference to a system of fixed rectangular axes of x, y, z, passing through the fixed point. Let ω be the angular velocity of the body about the instantaneous axis, and let ω_x, ω_y, ω_z, be the components of the angular velocity about the axes of coordinates.

Then $$\cos\lambda = \frac{\omega_x}{\omega}, \quad \cos\mu = \frac{\omega_y}{\omega}, \quad \cos\nu = \frac{\omega_z}{\omega},$$

and consequently, differentiating with respect to t, we have

$$-\sin\lambda \frac{d\lambda}{dt} = \frac{1}{\omega^2}\left(\omega \frac{d\omega_x}{dt} - \omega_x \frac{d\omega}{dt}\right),$$

$$-\sin\mu \frac{d\mu}{dt} = \frac{1}{\omega^2}\left(\omega \frac{d\omega_y}{dt} - \omega_y \frac{d\omega}{dt}\right),$$

$$-\sin\nu \frac{d\nu}{dt} = \frac{1}{\omega^2}\left(\omega \frac{d\omega_z}{dt} - \omega_z \frac{d\omega}{dt}\right).$$

Let r be a given length measured along the instantaneous axis from the fixed point. Then

$$x = r\cos\lambda, \quad y = r\cos\mu, \quad z = r\cos\nu,$$

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and therefore

$$\frac{dx}{dt}=-r\sin\lambda\frac{d\lambda}{dt},\quad \frac{dy}{dt}=-r\sin\mu\frac{d\mu}{dt},\quad \frac{dz}{dt}=-r\sin\nu\frac{d\nu}{dt},$$

whence

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}=r^2\left(\sin^2\lambda\frac{d\lambda^2}{dt^2}+\sin^2\mu\frac{d\mu^2}{dt^2}+\sin^2\nu\frac{d\nu^2}{dt^2}\right).$$

Let Ω be the angular velocity of the instantaneous axis in space: then

$$\Omega^2r^2=\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}.$$

From the above relations we see that

$$\omega^4\Omega^2=\left(\omega\frac{d\omega_x}{dt}-\omega_x\frac{d\omega}{dt}\right)^2+\left(\omega\frac{d\omega_y}{dt}-\omega_y\frac{d\omega}{dt}\right)^2+\left(\omega\frac{d\omega_z}{dt}-\omega_z\frac{d\omega}{dt}\right)^2$$

$$=\omega^2\left(\frac{d\omega_x^2}{dt^2}+\frac{d\omega_y^2}{dt^2}+\frac{d\omega^2}{dt^2}\right)$$

$$+(\omega_x^2+\omega^2+\omega_z^2)\frac{d\omega^2}{dt^2}$$

$$-2\omega\frac{d\omega}{dt}\left(\omega_x\frac{d\omega_x}{dt}+\omega_y\frac{d\omega_y}{dt}+\omega_z\frac{d\omega_z}{dt}\right).$$

But $$\omega_x^2+\omega_y^2+\omega_z^2=\omega^2,$$

and therefore $$\omega_x\frac{d\omega_x}{dt}+\omega_y\frac{d\omega_y}{dt}+\omega_z\frac{d\omega_z}{dt}=\omega\frac{d\omega}{dt}:$$

hence $$\omega^2\Omega^2=\frac{d\omega_x^2}{dt^2}+\frac{d\omega_y^2}{dt^2}+\frac{d\omega^2}{dt^2}-\frac{d\omega^2}{dt^2}.$$

Imagine the axes of coordinates to coincide with the principal axes of the body at the end of the time t, and let $\omega_1, \omega_2, \omega_3$, be the corresponding values of $\omega_x, \omega_y, \omega_z$: then, as is well known, $\frac{d\omega_x}{dt}, \frac{d\omega_y}{dt}, \frac{d\omega_z}{dt}$, are equal respectively to $\frac{d\omega_1}{dt}, \frac{d\omega_2}{dt}, \frac{d\omega_3}{dt}$. Thus

$$\omega^2\Omega^2=\frac{d\omega_1^2}{dt^2}+\frac{d\omega_2^2}{dt^2}+\frac{d\omega_3^2}{dt^2}-\frac{d\omega^2}{dt^2}.$$

Let a, b, c, be the moments of inertia about the principal axes through the fixed point: then, by Euler's formulæ,

$$a\frac{d\omega_1}{dt}=(b-c)\omega_2\omega_3,\quad b\frac{d\omega_2}{dt}=(c-a)\omega_3\omega_1,\quad c\frac{d\omega_3}{dt}=(a-b)\omega_1\omega_2:$$

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hence

$$\omega^2\Omega^2 = \left(\frac{b-c}{a}\right)^2 \omega_2^2\omega_3^2 + \left(\frac{c-a}{b}\right)^2 \omega_3^2\omega_1^2 + \left(\frac{a-b}{c}\right)^2 \omega_1^2\omega_2^2 - \frac{d\omega^2}{dt^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, f and g being arbitrary constants, we have, as is well known, as consequences from Euler's equations,

$$a\omega_1^2 + b\omega_2^2 + c\omega_3^2 = f \quad \ldots\ldots\ldots\ldots\ldots (2),$$

and

$$a^2\omega_1^2 + b^2\omega_2^2 + c^2\omega_3^2 = g^2 \quad \ldots\ldots\ldots\ldots\ldots (3).$$

Also

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega^2 \quad \ldots\ldots\ldots\ldots\ldots (4).$$

Let

$$\left.\begin{aligned} \frac{f(b+c)-g^2}{bc} &= \lambda_1 \\ \frac{f(c+a)-g^2}{ca} &= \lambda_2 \\ \frac{f(a+b)-g^2}{ab} &= \lambda_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (5):$$

then, from the equations (2), (3), (4), we obtain

$$\left.\begin{aligned} \omega_1^2 &= \frac{bc}{(c-a)(a-b)}(\lambda_1 - \omega^2) \\ \omega_2^2 &= \frac{ca}{(a-b)(b-c)}(\lambda_2 - \omega^2) \\ \omega_3^2 &= \frac{ab}{(b-c)(c-a)}(\lambda_3 - \omega^2) \end{aligned}\right\} \quad \ldots\ldots\ldots (6).$$

From the equations (6), combined with Euler's equations, we obtain, by the usual method, the known relation

$$\omega^2 \frac{d\omega^2}{dt^2} = (\lambda_1 - \omega^2)(\lambda_2 - \omega^2)(\lambda_3 - \omega^2) \quad \ldots\ldots\ldots (7).$$

From the equations (1), (6), and (7), we see that

$$\begin{aligned} \omega^2\Omega^2 = &\frac{bc}{(c-a)(a-b)}(\lambda_2 - \omega^2)(\lambda_3 - \omega^2) \\ &+ \frac{ca}{(a-b)(b-c)}(\lambda_3 - \omega^2)(\lambda_1 - \omega^2) \\ &+ \frac{ab}{(b-c)(c-a)}(\lambda_1 - \omega^2)(\lambda_2 - \omega^2) \\ &- \frac{1}{\omega^2}(\lambda_1 - \omega^2)(\lambda_2 - \omega^2)(\lambda_3 - \omega^2) \quad \ldots\ldots\ldots (8). \end{aligned}$$

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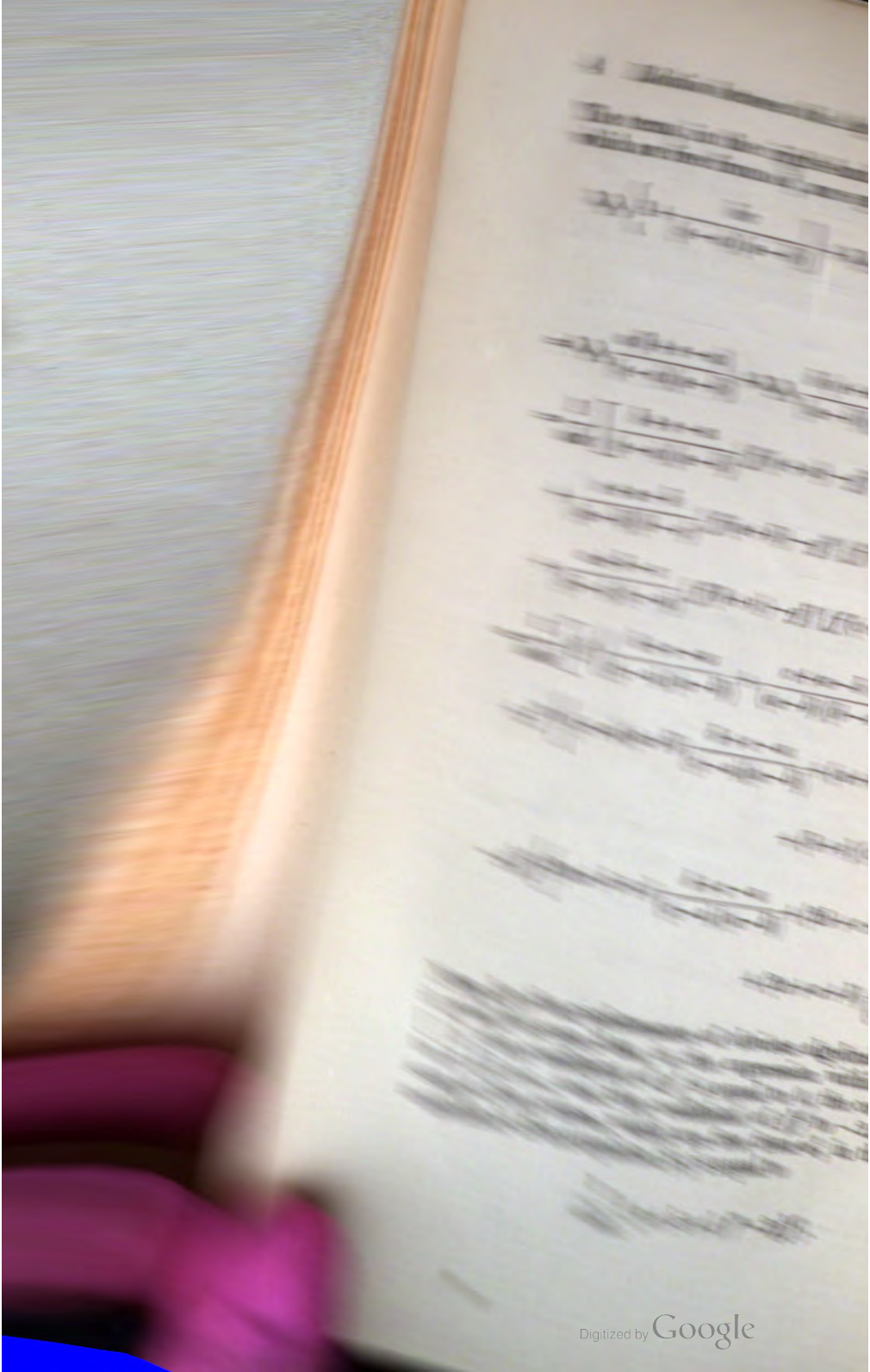
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Again, in the right-hand member of the equation (8), the coefficient of ω^2 is equal to

$$-\frac{bc(\lambda_2+\lambda_3)}{(c-a)(a-b)}-\frac{ca(\lambda_3+\lambda_1)}{(a-b)(b-c)}-\frac{ab(\lambda_1+\lambda_2)}{(b-c)(c-a)}-(\lambda_1+\lambda_2+\lambda_3)$$

$$=-\frac{bc}{(c-a)(a-b)}\left\{\frac{f(c+a)-g^2}{ca}+\frac{f(a+b)-g^2}{ab}\right\}$$

$$-\frac{ca}{(a-b)(b-c)}\left\{\frac{f(a+b)-g^2}{ab}+\frac{f(b+c)-g^2}{bc}\right\}$$

$$-\frac{ab}{(b-c)(c-a)}\left\{\frac{f(b+c)-g^2}{bc}+\frac{f(c+a)-g^2}{ca}\right\}$$

$$-\frac{f(b+c)-g^2}{bc}-\frac{f(c+a)-g^2}{ca}-\frac{f(a+b)-g^2}{ab}.$$

But, by obvious algebraical operations, it will be found that in this expression the coefficients of g^2 and f are both zero: hence, in the right-hand member of the equation (8), the coefficient of ω^2 is zero.

Again, in the right-hand member of the equation (8), the coefficient of ω^4 is equal to

$$\frac{bc}{(c-a)(a-b)}+\frac{ca}{(a-b)(b-c)}+\frac{ab}{(b-c)(c-a)}+1=0.$$

Finally, the coefficient of $\frac{1}{\omega^2}$ in the right-hand member of the equation (8) is equal to $-\lambda_1\lambda_2\lambda_3$.

Hence we have, for the permanent relation between ω^2 and Ω^2, the equation

$$\omega^2\Omega^2=\frac{f}{abc}\{(a+b+c)f-2g^2\}$$

$$-\frac{1}{a^2b^2c^2\omega^2}\{f(b+c)-g^2\}\{f(c+a)-g^2\}\{f(a+b)-g^2\}\ldots(9).$$

Let us suppose that of the quantities a, b, c, no two are equal, and that they are in the descending order of magnitude. Then, from the formulæ (6), it is evident that ω^2 is not $<\lambda_1$, ω^2 is not $>\lambda_2$, and ω^2 is not $<\lambda_3$; and that consequently λ_1 is not $>\lambda_2$ and λ_3 is not $>\lambda_2$.

Thus we see, by (6), that the greatest value of ω_1^2 is equal to

$$\frac{bc(\omega^2-\lambda_1)}{(a-b)(a-c)},$$

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when ω^2 is greatest; that is, that the greatest value of ω_1^2 is equal to

$$\frac{bc}{(a-b)(a-c)}(\lambda_2-\lambda_1)=\frac{g^2-cf}{a(a-c)}.$$

Since ω_1^2 is necessarily positive, this result shews that g^2 is not $<cf$.

Again, the greatest value of ω_3^2 is equal to

$$\frac{ab}{(a-c)(b-c)}(\omega^2-\lambda_3),$$

when ω^2 is greatest, *i.e.* when $\omega^2=\lambda_2$: thus the greatest value of ω_3^2 is equal to

$$\frac{ab}{(a-c)(b-c)}(\lambda_2-\lambda_3)=\frac{af-g^2}{c(a-c)},$$

and therefore g^2 is not greater than af.

Again, the greatest value of ω_2^2 is equal to

$$\frac{ca}{(a-b)(b-c)}(\lambda_2-\omega^2),$$

when ω^2 is least: thus the greatest value of ω_2^2 is equal to one or other of the values

$$\frac{ca}{(a-b)(b-c)}(\lambda_2-\lambda_1)=\frac{g^2-cf}{b(b-c)},$$

or

$$\frac{ca}{(a-b)(b-c)}(\lambda_2-\lambda_3)=\frac{af-g^2}{b(a-b)}.$$

Again,

$$\lambda_3-\lambda_1=\frac{a-c}{abc}(g^2-bf):$$

hence λ_3 is not $<\lambda_1$, if g^2 is not $<bf$; and λ_3 is not $>\lambda_1$, if g^2 is not $>bf$.

Suppose no two of the three quantities λ_1, λ_2, λ_3, to be equal: then, as is evident by the equation (7), no one of the quantities $\lambda_1-\omega^2$, $\lambda_2-\omega^2$, $\lambda_3-\omega^2$, can change sign: thus λ_2 is the greatest value of ω^2 and the greater of the two quantities λ_3, λ_1, is the least value of ω^2. Suppose that in the following investigations λ_2 is not less than λ_1, which if equivalent to the hypothesis that g^2 is not less

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than bf; it follows therefore that the greatest value of ω_2^2 is equal to

$$\frac{af-g^2}{b(a-b)}.$$

Next, suppose that $\lambda_1=\lambda_2$: then $g^2=cf$, which is not admissible since, by our supposition, g^2 is not less than bf.

If $\lambda_1=\lambda_3$, then $g^2=bf$, and the equation (7) becomes

$$\omega^2\frac{d\omega^2}{dt^2}=(\lambda_1-\omega^2)^2(\lambda_2-\omega^2).$$

Thus λ_3 is the greatest value of ω^2 and $\lambda_1=\lambda_3$ is the least value of ω^2.

If $\lambda_3=\lambda_2$, then

$$\omega_2^2=\frac{ca}{(a-b)(b-c)}(\lambda_2-\omega^2),$$

$$\omega_3^2=\frac{ab}{(b-c)(a-c)}(\omega^2-\lambda_2),$$

and consequently $\omega^2=\lambda_2$, $\omega_2^2=0$, $\omega_3^2=0$, whence also $\omega_1^2=\omega^2=\frac{f}{a}$. This case may be excluded from our consideration, since the instantaneous axis degenerates into a permanent one, viz. a principal axis of the body.

Thus, under all circumstances which affect our investigation, the greatest value of ω^2 is equal to λ_2 and the least to λ_3.

Thus we see that, when ω^2 is a maximum, *i.e.*, equal to λ_2,

$$\cos^2\lambda=\frac{\omega_1^2}{\omega^2}=\frac{bc}{(a-b)(a-c)}\cdot\frac{\lambda_2-\lambda_1}{\lambda_2}$$

$$=\frac{c}{a-c}\cdot\frac{g^2-cf}{f(c+a)-g^2},$$

$$\cos^2\mu=\frac{\omega_2^2}{\omega^2}=0,$$

$$\cos^2\nu=\frac{\omega_3^2}{\omega^2}=\frac{ab}{(a-c)(b-c)}\cdot\frac{\lambda_2-\lambda_3}{\lambda_2}$$

$$=\frac{a}{a-c}\cdot\frac{af-g^2}{f(c+a)-g^2}.$$

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Again, when ω^2 is a minimum, *i.e.* equal to λ_2,

$$\cos^2\lambda = \frac{\omega_1^2}{\omega^2} = \frac{bc}{(a-c)(a-b)} \cdot \frac{\lambda_2 - \lambda_1}{\lambda_2}$$

$$= \frac{b}{a-b} \cdot \frac{g^2 - bf}{f(a+b) - g^2},$$

$$\cos^2\mu = \frac{\omega_2^2}{\omega^2} = \frac{ca}{(a-b)(b-c)} \cdot \frac{\lambda_2 - \lambda_3}{\lambda_2}$$

$$= \frac{a}{a-b} \cdot \frac{af - g^2}{f(a+b) - g^2},$$

$$\cos^2\nu = \frac{\omega_3^2}{\omega^2} = 0.$$

To determine the value of Ω^2, when ω^2 is a maximum, we have only to substitute λ_2 for ω^2 in the equation (9), and, after a few obvious reductions, we shall get

$$\Omega^2 = \frac{(fa - g^2)(g^2 - fc)}{b^2\{f(c+a) - g^2\}}.$$

Similarly, when ω^2 is a minimum, the corresponding value of Ω^2 is given by the equation

$$\Omega^2 = \frac{(fa - g^2)(g^2 - fb)}{c^2\{f(a+b) - g^2\}}.$$

We proceed now to investigate the maximum and minimum values of Ω^2.

From the equation (9) we have

$$\omega^6 \frac{d(\Omega^2)}{d(\omega^2)} = \frac{2}{a^2b^2c^2}\{f(b+c) - g^2\}\{f(c+a) - g^2\}\{f(a+b) - g^2\}$$

$$- \frac{f\omega^2}{abc}\{(a+b+c)f - 2g^2\} \ldots\ldots\ldots\ldots (10).$$

Assume the expression for $\frac{d(\Omega^2)}{d(\omega^2)}$ to be zero: then

$$\omega^2 = \frac{2}{abc} \cdot \frac{\{f(b+c) - g^2\}\{f(c+a) - g^2\}\{f(a+b) - g^2\}}{(a+b+c)f^2 - 2fg^2} \ldots (11);$$

and therefore, by (9),

$$\omega^2\Omega^2 = \frac{f}{2abc}\{(a+b+c)f - 2g^2\} \ldots\ldots\ldots (12).$$

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That $\omega^2\Omega^2$ may be possible, $2g^2$ must not be greater than $(a+b+c)f$.

Again, since, as has been shewn above, g^2 is not greater than fa, it is less than either $f(a+b)$ or $f(c+a)$, and therefore, by (11), g^2 must not be greater than $f(b+c)$.

Again, since λ_2 is the greatest value of ω^2, we see, by referring to (11), that

$$\frac{2}{abc}\cdot\frac{\{f(b+c)-g^2\}\{f(c+a)-g^2\}\{f(a+b)-g^2\}}{(a+b+c)f^2-2fg^2} \text{ is not} > \frac{f(c+a)-g^2}{ca},$$

and therefore, since $f(c+a)-g^2$ and $(a+b+c)f-2g^2$ are neither of them supposed to be negative,

$$2\{f(b+c)-g^2\}\{f(a+b)-g^2\} \text{ not} > b\{(a+b+c)f^2-2fg^2\},$$

whence, by obvious simplifications,

$$\left(a+b+c-\frac{2g^2}{f}\right)^2 \text{ not} > (a+b-c)(a-b-c)\ldots(13).$$

Again, the least value of ω^2, as has been shewn above, is equal to

$$\lambda_2=\frac{f(a+b)-g^2}{ab}:$$

hence, by referring to (11) we see that

$$\frac{2}{abc}\cdot\frac{\{f(b+c)-g^2\}\{f(c+a)-g^2\}\{f(a+b)-g^2\}}{(a+b+c)f^2-2fg^2} \text{ not} < \frac{f(a+b)-g^2}{ab},$$

and therefore, by obvious reductions,

$$\left(a+b+c-\frac{2g^2}{f}\right)^2 \text{ not} < (b+c-a)(b-c-a)\ldots(14).$$

From (13) and (14) it is evident that

$$(a+b-c)(a-b-c) \text{ is not} < (b+c-a)(b-c-a),$$

$$0 \text{ not} < (b+c-a)\{(a+b-c)+(b-c-a)\},$$

$$0 \text{ not} < (b+c-a)(b-c),$$

and therefore a is not $< b+c$.

Thus we see that, unless a be not less than $b+c$, a maximum or minimum value of Ω^2 cannot be obtained by equating to zero the expression for $\frac{d(\Omega^2)}{d(\omega^2)}$.

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If $\frac{d(\Omega^2)}{d(\omega^2)} = 0$, then, from (10),

$$\omega^6 \frac{d^2(\Omega^2)}{\{d(\omega^2)\}^2} = -\frac{f}{abc}\{(a+b+c)f - 2g^2\} :$$

but, as has been shewn above, $(a+b+c)f$ is not less than $2g^2$. Thus, when $2g^2$ is less than $(a+b+c)f$, the value of ω^2 given by (11) corresponds to a maximum value of Ω^2.

Let us now consider especially the case when $(a+b+c)f = 2g^2$. Then, by (14), we see that

$$0 \text{ not} < (a-b-c)(c+a-b),$$

$$0 \text{ not} < a-b-c,$$

$$b+c \text{ not} < a.$$

But, as has been shewn above, a is not $< b+c$. Hence $a = b+c$, and therefore, $(a+b+c)f$ being equal to $2g^2$, we have $g^2 = af$. Hence, from the equations (2), (3), (4), we easily ascertain that $\omega_2^2 = 0$, $\omega_3^2 = 0$, and $\omega_1^2 = \omega^2 = \frac{f}{a}$. The instantaneous axis therefore coincides with a principal axis of the body, and is therefore immoveable. The relation

$$(a+b+c)f = 2g^2$$

may therefore be excluded from our investigations.

If the maximum value of Ω^2 above obtained correspond to a maximum value of ω^2, then, as will be seen by referring to (13),

$$\left(a+b+c-\frac{2g^2}{f}\right)^2 = (a+b-c)(a-b-c) \ldots (15).$$

If the maximum value of Ω^2 above obtained correspond to a minimum value of ω^2, then, by (14),

$$\left(a+b+c-\frac{2g^2}{f}\right)^2 = (a-b-c)(c+a-b) \ldots . (16).$$

Again, from the equation (9), we have

$$\omega^6 \frac{d(\Omega^2)}{dt} = \frac{d(\omega^2)}{dt}\left[\frac{2}{a^2b^2c^2}\{f(b+c)-g^2\}\{f(c+a)-g^2\}\{f(a+b)-g^2\}\right.$$

$$\left. - \frac{f\omega^2}{abc}\{(a+b+c)f - 2g^2\}\right] \ldots\ldots\ldots (17).$$

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Suppose next that the value of ω^2, derived from $\frac{d(\Omega^2)}{d(\omega^2)}=0$, not to be a maximum value of ω^2: then evidently, when $\frac{d(\omega^2)}{dt}$ changes sign from + to −, the expression within the brackets does not change sign: thus we see that when $\frac{d(\omega^2)}{dt}$ changes sign from + to −, $\frac{d(\Omega^2)}{dt}$ changes sign from + to − or from − to +. Consequently the maximum value of ω^2 corresponds to a maximum or minimum value of Ω^2. Let us see whether Ω^2 is a maximum or a minimum.

Substituting for ω^2 its maximum value, viz. λ_2, the expression within the brackets [ ] in the equation (17) becomes

$$\frac{f(c+a)-g^2}{2a^2b^2c^2}\left[\{f(a+b+c)-2g^2\}^2-f^2(a+b-c)(a-b-c)\right],$$

which expression, since it is not zero, is, in virtue of (13), negative. Consequently the maximum value of ω^2 corresponds to a minimum value of Ω^2.

Suppose next that the value of ω^2, derived from $\frac{d(\Omega^2)}{d(\omega^2)}=0$, not to be a minimum value of ω^2: then evidently the minimum value of ω^2 corresponds to a maximum or minimum value of Ω^2. Let us see whether Ω^2 is a maximum or a minimum.

Substituting for ω^2 its minimum value, viz. λ_3, the expression within the brackets [ ] in the equation (17) becomes

$$\frac{f(a+b)-g^2}{2a^2b^2c^2}\left[\{f(a+b+c)-2g^2\}^2-f^2(a-b-c)(c+a-b)\right],$$

which expression, since it is not zero, is, in virtue of (14), positive. Consequently the minimum value of ω^2 corresponds to a minimum value of Ω^2.

From the equations (6) and (11) we may easily infer the following expressions for $\cos\lambda$, $\cos\mu$, $\cos\nu$, viz.

$$\left.\begin{aligned}
\cos^2\lambda&=\frac{\omega_1^2}{\omega^2}=\frac{bc}{4(a-b)(a-c)}\cdot\frac{\{f(a+b+c)-2g^2\}^2+f^2(a+b-c)(a-b+c)}{\{f(c+a)-g^2\}\{f(a+b)-g^2\}}\\
\cos^2\mu&=\frac{\omega_2^2}{\omega^2}=\frac{ca}{4(b-c)(b-a)}\cdot\frac{\{f(a+b+c)-2g^2\}^2+f^2(b+c-a)(b-c+a)}{\{f(a+b)-g^2\}\{f(b+c)-g^2\}}\\
\cos^2\nu&=\frac{\omega_3^2}{\omega^2}=\frac{ab}{4(c-a)(c-b)}\cdot\frac{\{f(a+b+c)-2g^2\}^2+f^2(c+a-b)(c-a+b)}{\{f(b+c)-g^2\}\{f(c+a)-g^2\}}
\end{aligned}\right\}$$

.....(18).

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the instantaneous axis has four positions of maximum mobility, the corresponding angular velocities of the body being neither maxima nor minima; two positions of minimum mobility, the corresponding angular velocities of the body being maxima; and two positions of minimum mobility, the corresponding angular velocities of the body being minima.

(2) If $f(b+c) > g^2$, $a > b+c$, and

$$\left(a+b+c-\frac{2g^2}{f}\right)^2 \text{ not} < (a+b-c)\,(a-b-c),$$

the instantaneous axis has two positions of maximum mobility, the corresponding angular velocities of the body being maxima; and two positions of minimum mobility, the corresponding angular velocities of the body being minima.

(3) If $f(b+c) > g^2$, $a > b+c$, and

$$\left(a+b+c-\frac{2g^2}{f}\right)^2 \text{ not} > (a-b-c)\,(c+a-b),$$

the instantaneous axis has two positions of maximum mobility, the corresponding angular velocities of the body being minima; and two positions of minimum mobility, the corresponding angular velocities of the body being maxima.

(4) If $f(b+c) > g^2$ and $a < b+c$, the instantaneous axis has two positions of maximum mobility, the corresponding angular velocities of the body being maxima; and two positions of minimum mobility, the corresponding angular velocities of the body being minima.

We have in the preceding researches fully investigated the circumstances of the general problem on the hypothesis that λ_3 is not $< \lambda_1$, which is equivalent to the supposition that g^2 is not $< bf$. Suppose next that λ_1 is not $< \lambda_3$, and that accordingly g^2 is not $> bf$. By obvious adaptations of the preceding formulæ to the new hypothesis, we should easily arrive at the following conclusions:

1. If $f(b+c) > g^2$, $a > b+c$, and $\left(a+b+c-\frac{2g^2}{f}\right)^2$ be less than $(a+b-c)\,(a-b-c)$, the instantaneous axis has four positions of maximum mobility, the corresponding angular velocities of the body being neither maxima nor minima; two positions of minimum mobility, the corresponding angular velocities of the body being maxima; and two

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positions of maximum mobility, the corresponding angular velocities of the body being minima.

2. If $f(b+c) > g^2$, $a > b+c$, and

$$\left(a+b+c-\frac{2g^2}{f}\right)^2 \text{ not} < (a+b-c)(a-b-c),$$

the instantaneous axis has two positions of maximum mobility, the corresponding angular velocities of the body being maxima; and two positions of minimum mobility, the corresponding angular velocities of the body being minima.

3. If $f(b+c) > g^2$ and $a < b+c$, the instantaneous axis has two positions of maximum mobility, the corresponding angular velocities of the body being maxima, and two positions of minimum mobility, the corresponding angular velocities of the body being minima.

If $2g^2$ be $> (a+b+c)f$, the value of $\omega^2\Omega^2$ in the equation (12) is impossible: consequently $\frac{d(\Omega^2)}{d(\omega^2)}$ cannot be equated to zero. The equation

$$\frac{d(\Omega^2)}{dt} = \frac{d(\Omega^2)}{d(\omega^2)} \cdot \frac{d(\omega^2)}{dt} = 0$$

shews that in this case the only maxima and minima of Ω^2 are contemporaneous with those of ω^2 as given by equating to zero the factor $\frac{d(\omega^2)}{dt}$.

December 16, 1869.

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ON THE QUARTIC SURFACES $(*\chi U, V, W)^2 = 0$.

(Continued from Vol. x., p. 34).

By Professor CAYLEY.

AMONG the surfaces of the form in question are included the reciprocals of several interesting surfaces of the orders 6, 8, 9, 10, and 12, viz.

Order 6, parabolic ring.

„ 8, elliptic ring.

„ 9, centro-surface of a paraboloid.

„ 10, parallel surface of a paraboloid.

„ „ envelope of planes through the points of an ellipsoid at right angles to the radius vectors from the centre.

„ 12, centro-surface of ellipsoid.

„ „ parallel surface of an ellipsoid.

I propose to consider these surfaces, not at present in any detail, but merely for the purpose of presenting them in connexion with each other and with the present theory. It will be convenient to use homogeneous equations, but for the metrical interpretation the coordinate W or w may be considered as equal to unity: I have not thought it necessary so to adjust the constants so that the equations shall be homogeneous in regard to the constants; this can of course be done without difficulty, and in many cases it would be analytically advantageous to make the change.

I take throughout (X, Y, Z, W) for the coordinates of a point on the quartic surface (so that (U, V, W) in the equation $(*\chi U, V, W)^2 = 0$ are to be considered as quadric functions of (X, Y, Z, W)), reserving (x, y, z, w) for the coordinates of a point on the reciprocal surface of the order 6, 8, 9, 10, or 12. The reciprocation is performed in regard to the imaginary sphere $x^2 + y^2 + z^2 + w^2 = 0$: the relation between the coordinates (X, Y, Z, W) and (x, y, z, w) is then $Xx + Yy + Zz + Ww = 0$, and the equation $(X, Y, Z, W)^4 = 0$ is the equation in point-coordinates of the quartic surface, or in line-coordinates of the reciprocal surface: and similarly the equation $(x, y, z, w)^n = 0$ is the equation in point-coordinates of the reciprocal surface, or in line-coordinates of the quartic surface.

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Parabolic ring, or envelope of a sphere of constant radius having its centre on a parabola.

Taking k for the radius of the sphere, and $z = 0$, $y^2 = 4ax$ for the equations of the parabola, then the coordinates of a point on the parabola are $a\theta^2$, $2a\theta$, 0; where θ is a variable parameter. The equation of the sphere therefore is

$$(x - a\theta^2 w)^2 + (y - 2a\theta w)^2 + z^2 - k^2 w^2 = 0,$$

and the ring is the envelope of this sphere.

The reciprocal of the sphere is

$$k^2 (X^2 + Y^2 + Z^2) - (a\theta^2 X + 2a\theta Y + W)^2 = 0;$$

writing this in the form

$$a\theta^2 X + 2a\theta Y + W + k \sqrt{(X^2 + Y^2 + Z^2)} = 0,$$

and taking the envelope in regard to θ, we have

$$X \{W + k \sqrt{(X^2 + Y^2 + Z^2)}\} - aY^2 = 0,$$

or, what is the same thing,

$$(aY^2 - XW)^2 - k^2 X^2 (X^2 + Y^2 + Z^2) = 0,$$

for the equation of the quartic surface. This has the line $X = 0$, $Y = 0$ for a tacnodal line, but I am not in possession of a theory enabling me thence to infer that the parabolic ring is of the order 6.

To shew that it is so, I revert to the equation of the variable sphere

$$(x - a\theta^2 w)^2 + (y - 2a\theta w)^2 + z^2 - k^2 w^2 = 0,$$

or what is the same thing

$$(A, B, C, D, E\mathfrak{X}\theta, 1)^4 = 0,$$

where

$$A = 3a^2 w^2,$$

$$B = 0,$$

$$C = a(2aw^2 - xw),$$

$$D = -3ayw,$$

$$E = 3(x^2 + y^2 + z^2 - k^2 w^2).$$

Then $I = 3a^2 w^2 I'$, $J = a^3 w^3 J'$, and the equation is $I'^3 - J'^2 = 0$, viz. this is

$$\{4x^2 + 3y^2 - 4axw + 4a^2 w^2 + 3(z^2 - k^2 w^2)\}^3$$

$$- \{(2aw - x)[8x^2 + 9y^2 + 4axw - 4a^2 w^2 + 9(z^2 - k^2 w^2)] - 27ay^2 w\}^2 = 0,$$

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or as this may also be written

$$\{4x^2 + 3y^2 - 4axw + 4a^2w^2 + 3(z^2 - k^2w^2)\}^2$$
$$- \{-8x^2 - 9xy^2 + 12ax^2w + 12a^2xw^2 - 8a^2w^3$$
$$- 9(x - 2aw)(z^2 - k^2w^2)\}^2 = 0.$$

Developing, the whole divides by 27, and the equation of the ring finally is

$$(y^2 - 4axw)^2\{y^2 + (x - aw)^2\}$$
$$+ \{3y^4 + y^2(2x^2 - 2axw + 20a^2w^2)$$
$$+ 8ax^3w + 8a^2x^2w^2 - 32a^3xw^3 + 16a^4w^4\}(z^2 - k^2w^2)$$
$$+ (3y^2 + x^2 + 8axw - 8a^2w^2)(z^2 - k^2w^2)^2$$
$$+ (z^2 - k^2w^2)^3 = 0.$$

Elliptic ring, or envelope of a sphere of constant radius having its centre on an ellipse.

Taking k for the radius of the sphere, and $z = 0$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for the equations of the ellipse, the coordinates of a point on the ellipse are $a\cos\theta$, $b\sin\theta$; hence the equation of the variable sphere is

$$(x - aw\cos\theta)^2 + (y - bw\sin\theta)^2 + z^2 - k^2w^2 = 0.$$

The reciprocal of this is

$$k^2(X^2 + Y^2 + Z^2) - (aX\cos\theta + bY\sin\theta + W)^2 = 0.$$

viz. writing this under the form

$$aX\cos\theta + bY\sin\theta + W + k\sqrt{(X^2 + Y^2 + Z^2)} = 0,$$

and taking the envelope in regard to θ, the equation of the reciprocal surface is

$$a^2X^2 + b^2Y^2 = \{W + k\sqrt{(X^2 + Y^2 + Z^2)}\}^2,$$

viz. this is

$$(a^2 - k^2)X^2 + (b^2 - k^2)Y^2 - k^2Z^2 - W^2 = 2kW\sqrt{(X^2 + Y^2 + Z^2)},$$

or $\{(a^2 - k^2)X^2 + (b^2 - k^2)Y^2 - k^2Z^2 - W^2\}^2 - 4k^2W^2(X^2 + Y^2 + Z^2) = 0,$

that is $\{(a^2 - k^2)X^2 + (b^2 - k^2)Y^2 - k^2Z^2\}^2$
$$- 2W^2\{(a^2 + k^2)X^2 + (b^2 + k^2)Y^2 + k^2Z^2\} + W^4 = 0,$$

which is a quartic surface having the nodal conic $W = 0$, $(a^2 - k^2)X^2 + (b^2 - k^2)Y^2 - k^2Z^2 = 0$. This singularity alone would only reduce the order of the reciprocal surface to 12;

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the reciprocal surface or elliptic ring is in fact (as I proceed to show) of the order 8.

For this purpose reverting to the equation

$$(x - aw\cos\theta)^2 + (y - bw\sin\theta)^2 + z^2 - k^2w^2 = 0,$$

this may be written

$$A\cos 2\theta + B\sin 2\theta + C\cos\theta + D\sin\theta + E = 0,$$

where

$$\begin{aligned} A &= (a^2 - b^2)w^2, \\ B &= 0, \\ C &= -4axw, \\ D &= -4byw, \\ E &= (a^2 + b^2)w^2 + 2(x^2 + y^2 + z^2 - k^2w^2), \end{aligned}$$

and the equation is

$$\begin{aligned} &\{12(A^2 + B^2) - 3(C^2 + D^2) + 4E^2\}^3 \\ &\quad - \{27A(C^2 - D^2) + 54BCD - [72(A^2 + B^2) + 9(C^2 + D^2)]E \\ &\quad + 8E^3\}^2 = 0, \end{aligned}$$

or say

$$\begin{aligned} &\{12A^2 - 3(C^2 + D^2) + 4E^2\}^3 \\ &\quad - \{27A(C^2 - D^2) - [72A^2 + 9(C^2 + D^2)]E + 8E^3\}^2 = 0. \end{aligned}$$

This is of the order 12, but it is easy to see that the terms in E^6 and $E^4(C^2 + D^2)$ disappear from the equation, all the other terms divide by w^4; and the equation is thus of the order 8.

The equation may be obtained somewhat differently as follows. The equation of the variable sphere is

$$(x - \alpha w)^2 + (y - \beta w)^2 + z^2 - k^2w^2 = 0,$$

where (α, β) vary subject to the condition $\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1$. We have therefore

$$x - \alpha w - \lambda\frac{\alpha w}{a^2} = 0,$$

$$y - \beta w - \lambda\frac{\beta w}{b^2} = 0,$$

and thence

$$\alpha w = \frac{a^2x}{a^2 + \lambda}, \quad x - \alpha w = -\frac{\lambda x}{a^2 + \lambda},$$

$$\beta w = \frac{b^2y}{b^2 + \lambda}, \quad y - \beta w = -\frac{\lambda y}{b^2 + \lambda}.$$

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Consequently

$$\frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2 - k^2w^2}{\lambda^2} = 0,$$

$$\frac{a^2x^2}{(a^2+\lambda)^2} + \frac{b^2y^2}{(b^2+\lambda)^2} - \quad w^2 \quad = 0,$$

from which λ is to be eliminated. The second equation may be replaced by

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2 - k^2w^2}{\lambda} - w^2 = 0,$$

which has the first for its derived equation in regard to λ. Hence, writing this last equation in the form

$$\omega^2 (a^2+\lambda)(b^2+\lambda)\lambda - (b^2+\lambda)\lambda x^2 - (a^2+\lambda)\lambda y^2 - (a^2+\lambda)(b^2+\lambda)(z^2 - k^2\omega^2) = 0,$$

we have to equate to zero the discriminant of this cubic function of λ. Calling the equation

$$(A, B, C, D)(\lambda, 1)^3 = 0,$$

we have

$$A = 3w^2,$$

$$B = (a^2+b^2)w^2 - x^2 - y^2 - (z^2 - k^2w^2),$$

$$C = a^2b^2w^2 - b^2x^2 - a^2y^2 - (a^2+b^2)(z^2 - k^2w^2),$$

$$D = -3a^2b^2(z^2 - k^2w^2).$$

The required equation then is

$$A^2D^2 + 4AC^3 + 4B^3D - 3B^2C^2 - 6ABCD = 0.$$

The developed equation (Salmon's *Conic Sections*, Ed. v., p. 325) is

$$(b^2x^2 + a^2y^2 - a^2b^2w^2)^2 \{(x - cw)^2 + y^2\} \{(x + cw)^2 + y^2\},$$

$$- \left\{ \begin{array}{l} 2b^2(a^2 - 2b^2)x^6 - 2(a^4 - a^2b^2 + 3b^4)x^4y^2 \\ \quad - 2(3a^4 - a^2b^2 + b^4)x^2y^4 + 2a^2(b^2 - 2a^2)y^6 \\ - b^2(6a^4 - 10a^2b^2 + 6b^4)x^4w^2 \\ \quad + (4a^6 - 6a^4b^2 - 6a^2b^4 + 4b^6)x^2y^2w^2 \\ \quad - a^2(6a^4 - 10a^2b^2 + 6b^4)y^4w^2 \\ + 2c^2(3a^4 - a^2b^2 + b^4)x^2w^4 - 2c^2(a^4 - 3a^2b^2 + 3b^4)y^2w^4 \\ \quad - 2a^2b^2c^2(a^2 + b^2)w^6 \end{array} \right\} (z^2 - k^2\omega^2)$$

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$$+\left\{\begin{array}{r}(a^4-6a^2b^2+6b^4)\,x^4+(6a^4-10a^2b^2+6b^4)\,x^2y^2\\ +(6a^4-6a^2b^2+b^4)\,y^4\\ -(2c^2(a^4-a^2b^2+3b^4)x^2w^2+2c^2(3a^4-a^2b^2+b^4)y^2w^2\\ +c^4(a^4+4a^2b^2+b^4)\,w^4\end{array}\right\}(z^2-k^2w^2)^2$$

$$+\left\{\begin{array}{r}(a^2-2b^2)\,x^2+(2a^2-b^2)\,y^2\\ +c^2(a^2+b^2)\,w^2\end{array}\right\}2c^2(z^2-k^2w^2)^2$$

$$+c^4(z^2-k^2w^2)^4=0.$$

I remark that the before-mentioned nodal conic $W=0$, $(a^2-k^2)X^2+(b^2-k^2)Y^2-k^2Z^2=0$ is the reciprocal of a quadric cone, which is a bitangent cone of the ring: this is a cone, vertex at the centre of the ring, and which is the envelope of the right cone, vertex the same point, circumscribed about the variable sphere which generates the ring.

Centro-surface of a paraboloid.

For the paraboloid $\frac{X^2}{a}+\frac{Y^2}{b}-2ZW=0$, it may be shewn that the centro-surface is the envelope of the quadric

$$\frac{ax^2}{(a+\theta)^2}+\frac{by^2}{(b+\theta)^2}-2zw-2\theta w^2=0.$$

The quartic surface is consequently the envelope of the quadric

$$\frac{(a+\theta)^2}{a}X^2+\frac{(b+\theta)^2}{b}Y^2+2\theta Z^2-2ZW=0,$$

viz. this is

$$\theta^2\left(\frac{X^2}{a}+\frac{Y^2}{b}\right)+2\theta(X^2+Y^2+Z^2)+aX^2+bY^2-2ZW=0.$$

Hence the quartic surface is

$$\left(\frac{X^2}{a}+\frac{Y^2}{b}\right)(aX^2+bY^2-2ZW)-(X^2+Y^2+Z^2)^2=0,$$

or, what is the same thing,

$$X^2Y^2(a-b)^2-2ZW(bX^2+aY^2)-2abZ^2(X^2+Y^2)-abZ^4=0.$$

This has four conic nodes; viz. considering the equations

$$\frac{X^2}{a}+\frac{Y^2}{b}=0,\ aX^2+bY^2-2ZW=0,\ X^2+Y^2+Z^2=0,$$

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these give the point $X=0$, $Y=0$, $Z=0$ four times; and four other points which are the nodes in question; the point $(X=0, Y=0, Z=0)$ is a singular point of a higher order; the reduction caused by these singularities should be $= 8+19$, so as to make the order of the surface of centres $= 9$; that is the reduction on account of the point $(X=0, Y=0, Z=0)$ must be $=19$; but it is not by any means obvious how this is so.

Parallel surface of the paraboloid.

This is given, Salmon's *Solid Geometry*, 2nd Edit., pp. 146 and 148, for the paraboloid $aX^2 + bY^2 + 2rZW = 0$, as the envelope of the quadric surface

$$\frac{\theta a x^2}{\theta a + 1} + \frac{\theta b y^2}{\theta b + 1} + 2\theta r z w - (\theta^2 r^2 + k^2) w^2 = 0.$$

The reciprocal quartic is thus the envelope of

$$\frac{\theta a + 1}{\theta a} X^2 + \frac{\theta b + 1}{\theta b} Y^2 + \frac{\theta^2 r^2 + k^2}{\theta^2 r^2} Z^2 + \frac{2}{\theta r} ZW = 0,$$

that is

$$(X^2 + Y^2 + Z^2) + \frac{1}{\theta}\left(\frac{X^2}{a} + \frac{Y^2}{b} + \frac{2}{r} ZW\right) + \frac{1}{\theta^2}\frac{k^2}{r^2} Z^2 = 0,$$

whence the equation is

$$4\frac{k^2}{r^2} Z^2 (X^2 + Y^2 + Z^2) - \left(\frac{X^2}{a} + \frac{Y^2}{b} + \frac{2}{r} ZW\right)^2 = 0,$$

viz. this is a quartic having the nodal line-pair $Z = 0$, $\frac{X^2}{a} + \frac{Y^2}{b} = 0$; and a further singularity at the point $X=0$, $Y=0$, $Z=0$. It would require some consideration to show that the order of the parallel surface is thence $= 10$, as it should be.

Envelope of the planes through the points of an ellipsoid at right angles to the radius vectors from the centre.

This is given in my paper "Sur la Surface, &c." in the *Annali di Matematica*, t. II., (1859), as the envelope of the quadric surface

$$\frac{x^2}{2 - \frac{\theta}{a^2}} + \frac{y^2}{2 - \frac{\theta}{b^2}} + \frac{z^2}{2 - \frac{\theta}{c^2}} - \theta w^2 = 0.$$

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The reciprocal quartic surface is thus the envelope of

$$\left(2-\frac{\theta}{a^2}\right)X^2+\left(2-\frac{\theta}{b^2}\right)Y^2+\left(2-\frac{\theta}{c^2}\right)Z^2-\frac{1}{\theta}W^2=0,$$

or, what is the same thing,

$$\theta\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)-2(X^2+Y^2+Z^2)+\frac{1}{\theta}W^2=0,$$

viz. this is

$$W^2\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)-(X^2+Y^2+Z^2)^2=0,$$

which is in fact the inverse surface

$$\left(\frac{X}{X^2+Y^2+Z^2}, \frac{Y}{X^2+Y^2+Z^2}, \frac{Z}{X^2+Y^2+Z^2} \text{ for } X, Y, Z\right)$$

of the ellipsoid $\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1$; this is obvious geometrically inasmuch as the reciprocal of the variable plane is the inverse of the point on the ellipsoid.

The quartic surface has the nodal conic

$$W=0,\ X^2+Y^2+Z^2=0;$$

and also the node $X=0$, $Y=0$, $Z=0$; there is consequently in the order of the reciprocal surface a reduction $24+2=26$, or the order of the reciprocal surface is $=10$.

Centro-surface of the ellipsoid.

Writing the equation of the ellipsoid in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-w^2=0$, the centro-surface is given as the envelope of the quadric surface

$$\frac{a^2x^2}{(\theta+a^2)^2}+\frac{b^2y^2}{(\theta+b^2)^2}+\frac{c^2z^2}{(\theta+c^2)^2}-w^2=0,$$

(Salmon, p. 400) and hence the reciprocal quartic surface is the envelope of

$$\left(a+\frac{\theta}{a}\right)^2X^2+\left(b+\frac{\theta}{b}\right)^2Y^2+\left(c+\frac{\theta}{c}\right)^2Z^2-W^2=0,$$

in regard to the variable parameter θ, viz. the equation is

$$\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)(a^2X^2+b^2Y^2+c^2Z^2-W^2)-(X^2+Y^2+Z^2)^2=0,$$

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(see Salmon, p. 144). It hence at once appears, that the quartic surface has 12 nodes, viz. these are the four angles of the fundamental tetrahedron $(XYZW)$, and the eight points

$$\begin{cases} \dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} + \dfrac{Z^2}{c^2} = 0, \\ X^2 + Y^2 + Z^2 = 0, \\ a^2X^2 + b^2Y^2 + c^2Z^2 - W^2 = 0, \end{cases}$$

or writing as it is convenient to do

$$(\alpha, \beta, \gamma) = (b^2 - c^2,\ c^2 - a^2,\ a^2 - b^2);$$

and therefore

$$\alpha + \beta + \gamma = 0,\quad a^2\alpha + b^2\beta + c^2\gamma = 0,\quad a^4\alpha + b^4\beta + c^4\gamma = -\alpha\beta\gamma;$$

these are the eight points

$$\frac{X^2}{W^2} = -\frac{a^2}{\beta\gamma},\quad \frac{Y^2}{W^2} = -\frac{b^2}{\gamma\alpha},\quad \frac{Z^2}{W^2} = -\frac{c^2}{\alpha\beta}.$$

the order of the reciprocal of the quartic surface is thus $36 - 2.12, = 12$, which is in fact the order of the surface of centres.

The equation of the centro-surface is given, Salmon, p. 151, and *Quart. Math. Jour.*, t. II., (1858), p. 220, in the form

$$(\alpha, \beta, \gamma)^6 (\xi, \eta, \zeta, \omega)^{12} = 0,$$

where ξ, η, ζ, ω stand for ax, by, cz, iw; it is therefore of the degree 18 in regard to a, b, c.

Parallel surface of the ellipsoid.

This is given, Salmon, p. 148, as the envelope of the quadric surface

$$\frac{x^2}{a^2 + \theta} + \frac{y^2}{b^2 + \theta} + \frac{z^2}{c^2 + \theta} - \left(1 + \frac{k^2}{\theta}\right)\omega^2 = 0.$$

The reciprocal quartic is thus the envelope of

$$(a^2 + \theta)\, X^2 + (b^2 + \theta)\, Y^2 + (c^2 + \theta)\, Z^2 - \frac{\theta W^2}{k^2 + \theta} = 0,$$

or writing $k^2 + \theta = \lambda$, this is

$$(a^2 - k^2 + \lambda)\, X^2 + (b^2 - k^2 + \lambda)\, Y^2 + (c^2 - k^2 + \lambda)\, Z^2 - \left(1 - \frac{k^2}{\lambda}\right) W^2 = 0,$$

or, what is the same thing,

$$\lambda^2 (X^2 + Y^2 + Z^2)$$
$$+ \lambda\left[(a^2 - k^2)\, X^2 + (b^2 - k^2)\, Y^2 + (c^2 - k^2)\, Z^2 - W^2\right] + k^2 W^2 = 0,$$

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whence the equation is

$$\{(a^2-k^2)X^2+(b^2-k^2)Y^2+(c^2-k^2)Z^2-W^2\}^2 - 4k^2W^2(X^2+Y^2+Z^2)=0,$$

viz. this is a quartic having the nodal conic

$$W=0,\ (a^2-k^2)X^2+(b^2-k^2)Y^2+(c^2-k^2)Z^2=0.$$

The order of the reciprocal or parallel surface is thus $36-24$, $=12$, as it should be. The nodal conic of the quartic surface is the reciprocal of a bitangent or node-couple quadric cone, vertex the centre, in the parallel surface: this cone is imaginary for the ellipsoid, but real for either of the hyperboloids, and its existence in the case of the hyperboloid is readily perceived.

Reverting to the equation

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}-\left(1+\frac{k^2}{\theta}\right)w^2=0,$$

or say

$$(a^2+\theta)(b^2+\theta)(c^2+\theta)(k^2+\theta)w^2$$
$$-x^2(b^2+\theta)(c^2+\theta)\theta-y^2(c^2+\theta)(a^2+\theta)\theta-z^2(a^2+\theta)(b^2+\theta)\theta=0.$$

This is $(A, B, C, D, E)(\theta, 1)^4=0,$

where putting for shortness

$$\alpha=a^2+b^2+c^2+k^2,$$
$$\beta=b^2c^2+c^2a^2+a^2b^2+k^2(a^2+b^2+c^2),$$
$$\gamma=a^2b^2c^2+k^2(b^2c^2+c^2a^2+a^2b^2),$$
$$\delta=a^2b^2c^2k^2,$$

and

$$p=x^2+y^2+z^2,$$
$$q=(b^2+c^2)x^2+(c^2+a^2)y^2+(a^2+b^2)z^2,$$
$$r=b^2c^2x^2+c^2a^2y^2+a^2b^2z^2,$$

we have

$$A=12w^2,$$
$$B=3\alpha w^2-3p,$$
$$C=2\beta w^2-2q,$$
$$D=3\gamma w^2-3r,$$
$$E=12\delta w^2.$$

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The equation of the parallel surface is of course

$$(AE - 4BD + 3C^2)^2 - 27(ACE - AD^2 - B^2E + 2BCD - C^3)^2 = 0.$$

It is remarked (Salmon, p. 148) that there is in the plane $z = 0$, a nodal conic $\frac{x^2}{a-c} + \frac{y^2}{b-c} - \left(1 + \frac{k^2}{c}\right) w^2 = 0$, the complete section being made up of this conic twice, and of the curve of the eighth order which is the parallel curve of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - w^2 = 0$; the like is of course the case as to the sections by the other two principal planes $x = 0$ and $y = 0$. For the section by the plane $w = 0$ (or plane infinity) we have at once $p^2r^2(4pr - q^2) = 0$, where observe that

$$q^2 - 4pr = \{(b^2 + c^2)x^2 + (c^2 + a^2)y^2 + (a^2 + b^2)z^2\}^2$$
$$- 4(x^2 + y^2 + z^2)(b^2c^2x^2 + c^2a^2y^2 + a^2b^2z^2)$$
$$= (1, 1, 1, -1, -1, -1)(*)(b^2 - c^2)x^2, (c^2 - a^2)y^2, (a^2 - b^2)z^2)^2$$
$$= \text{norm.}\{x\sqrt{(b^2 - c^2)} + y\sqrt{(c^2 - a^2)} + z\sqrt{(a^2 - b^2)}\}.$$

The section is thus made up of two conics, each twice, and of four right lines; viz. the conics are $x^2 + y^2 + z^2 = 0$, the circle at infinity and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$, the section at infinity of the ellipsoid. And the lines are

$$x\sqrt{(b^2 - c^2)} \pm y\sqrt{(c^2 - a^2)} \pm z\sqrt{(a^2 - b^2)} = 0.$$

viz. these are the common tangents of the two conics. The circle at infinity is a nodal conic on the surface, which has thus 4 nodal conics.

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GENERAL SOLUTION AND EXTENSION OF THE PROBLEM OF THE 15 SCHOOL GIRLS.

By Andrew Frost.

THIS problem, generally stated, requires that a certain number of girls walk so placed in groups of 3 or more that no 2 girls appear together on any of the days for which such different arrangements are possible.

The following general solution may be interesting to readers of the *Journal* as by it the particular case investigated in Vol. VIII., p. 236, may be extended to cases in which the method adopted by Mr. Power, and previously in the *Lady's Magazine*, would be too complicated.

Let p denote one of 15 girls, and a_1a_2, b_1b_2, ... g_1g_2, one of the various ways in which the remaining 14 girls walk 2 and 2 with p on the 7 days for which alone the requirements of the problem can be satisfied.

We have to investigate and place below the 7 triads

$$pa_1a_2,\ pb_1b_2,\ \ldots pg_1g_2,$$

7 different sets of 4 triads composed of $a_1a_2 \ldots g_1g_2$ in such a manner that no 2 appear together in more than one triad. Let a be the general representative of a_1 and a_2 and so on. Now the 7 letters a, b, c, ... g, can be arranged variously in 7 triads, so that no 2 letters come together in 2 of them.

Let us take one such arrangement as

$$abc,\ ade,\ afg,\ bdf,\ beg,\ cdg,\ cef,$$

and call these fundamental triads.

Each of which, as abc, represents 4 triads, such as

$$a_1b_1c_1,\ a_1b_2c_2,\ a_2b_1c_2,\ a_2b_2c_1,$$

which is one of the various arrangements of the subscripts.

Hence the 5 triads arranged for each of the 7 days may be thus written in general form:

pa_1a_2	pb_1b_2	pc_1c_2	pd_1d_2	pe_1e_2	pf_1f_2	pg_1g_2
bdf	*cdg*	*ade*	*abc*	*abc*	*abc*	*abc*
beg	*cef*	*afg*	*afg*	*afg*	*ade*	*ade*
cdg	*ade*	*bdf*	*beg*	*bdf*	*beg*	*bdf*
cef	*afg*	*beg*	*cef*	*cdg*	*cdg*	*cef*

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Here each of the 7 fundamental triads occurring 4 times, it remains only to investigate arrangements of subscripts for each consistently with the requirements of the problem.

Commencing with the triad bdf, we may regard it as representing the 4 triads

$$b_1d_1f_1,\ b_1d_2f_2,\ b_2d_1f_2,\ b_2d_2f_1.$$

If we write these subscripts in the 4 vertical columns in which bdf appears, and observe, that as b_1 occurs in $b_1d_1f_1$ of the 1st column, the other b in the same column must have the subscript 2, and so throughout, we have

pa_1a_2	pb_1b_2	pc_1c_2	pd_1d_2	pe_1e_2	pf_1f_2	pg_1g_2
$b_1d_1f_1$	cdg	ad_1e	abc	ab_1c	abc	ab_1c
b_2eg	cef	af_1g	afg	af_1g	ade	ad_1e
cd_2g	ade	$b_1d_2f_2$	beg	$b_2d_1f_2$	beg	$b_2d_2f_1$
cef_2	afg	b_2eg	cef	cd_2g	cdg	cef_2

Again, if we take the triad cdg we find

$$cd_2g,\ cdg,\ cd_2g,\ cdg,$$

their remaining subscripts may be thus written

$$c_1d_2g_1,\ c_1d_1g_2,\ c_2d_2g_1,\ c_2d_1g_1,$$

thus we have

pa_1a_2	pb_1b_2	pc_1c_2	pd_1d_2	pe_1e_2	pf_1f_2	pg_1g_2
$b_1d_1f_1$	$c_1d_1g_2$	ad_1e	abc	ab_1c_1	abc_1	ab_1c
b_2eg_2	c_2ef	af_1g	afg	af_1g_1	ad_2e	ad_1e
$c_1d_2g_1$	ad_2e	$b_1d_2f_2$	beg	$b_2d_1f_2$	beg_2	$b_2d_2f_1$
c_2ef_2	afg_1	b_2eg	cef	$c_2d_2g_2$	$c_2d_1g_1$	cef_2

Again, if we take the triad beg, we find

$$b_2eg_2,\ b_2eg,\ beg,\ beg_2,$$

the remaining subscripts may be thus written

$$b_2e_1g_2,\ b_2e_2g_1,\ b_1e_1g_1,\ b_1e_2g_2,$$

thus we have

pa_1a_2	pb_1b_2	pc_1c_2	pd_1d_2	pe_1e_2	pf_1f_2	pg_1g_2
$b_1d_1f_1$	$c_1d_1g_2$	ad_1e_1	ab_2c	ab_1c_1	ab_2c_1	ab_1c
$b_2e_1g_2$	c_2ef	af_1g_2	afg_2	af_1g_1	ad_2e_1	ad_1e
$c_1d_2g_1$	ad_2e	$b_1d_2f_2$	$b_1e_1g_1$	$b_2d_1f_2$	$b_1e_2g_2$	$b_2d_2f_1$
$c_2e_2g_2$	afg_1	$b_2e_2g_1$	ce_2f	$c_2d_2g_2$	$c_2d_1g_1$	cef_2

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The triads of the form *ade* stand thus

$$ad_2e_2,\ ad_1e_1,\ ad_2e_1,\ ad_1e_2,$$

the subscripts of which may be thus written

$$a_1d_2e_2,\ a_1d_1e_1,\ a_2d_2e_1,\ a_2d_1e_2,$$

thus we have

pa_1a_2	pb_1b_2	pc_1c_2	pd_1d_2	pe_1e_2	pf_1f_2	pg_1g_2
$b_1d_1f_1$	$c_1d_1g_2$	$a_1d_1e_1$	ab_2c	ab_1c_1	$a_1b_2c_1$	a_1b_1c
$b_2e_1g_2$	c_2e_1f	$a_2f_1g_2$	afg_2	af_1g_1	$a_2d_2e_1$	$a_2d_1e_2$
$c_1d_2g_1$	$a_1d_2e_2$	$b_1d_2f_2$	$b_1e_1g_1$	$b_2d_1f_2$	$b_1e_2g_2$	$b_2d_2f_1$
$c_2e_2f_2$	a_2fg_1	$b_2e_2g_1$	ce_2f	$c_2d_2g_2$	$c_2d_1g_1$	ce_1f_2

The triads of the form *afg* stand thus

$$a_2fg_1,\ a_2f_1g_2,\ afg_2,\ af_1g_1,$$

their remaining subscripts must be thus written

$$a_2f_2g_1,\ a_2f_1g_2,\ a_1f_2g_2,\ a_1f_1g_1,$$

and we have

pa_1a_2	pb_1b_2	pc_1c_2	pd_1d_2	pe_1e_2	pf_1f_2	pg_1g_2
$b_1d_1f_1$	$c_1d_1g_2$	$a_1d_1e_1$	a_2b_2c	$a_2b_1c_1$	$a_1b_2c_1$	a_1b_1c
$b_2e_1g_2$	$c_2e_1f_1$	$a_2f_1g_2$	$a_1f_2g_2$	$a_1f_1g_1$	a_2d_2e	$a_2d_1e_2$
$c_1d_2g_1$	$a_1d_2e_2$	$b_1d_2f_2$	$b_1e_1g_1$	$b_2d_1f_2$	$b_1e_2g_2$	$b_2d_2f_1$
$c_2e_2f_2$	$a_2f_2g_1$	$b_2c_2e_1$	ce_2f_1	$c_2d_2g_2$	$c_2d_1g_1$	ce_1f_2

The triads of the form *abc* stand thus

$$a_2b_2c,\ a_2b_1c_1,\ a_1b_2c_1,\ a_1b_1c,$$

in which the subscript of c in the first and fourth triad must be 2, thus we have

pa_1a_2	pb_1b_2	pc_1c_2	pd_1d_2	pe_1e_2	pf_1f_2	pg_1g_2
$b_1d_1f_1$	$c_1d_1g_2$	$a_1d_1e_1$	$a_2b_2c_2$	$a_2b_1c_1$	$a_1b_2c_1$	$a_1b_1c_2$
$b_2e_1g_2$	$c_2e_1f_1$	$a_2f_1g_2$	$a_1f_2g_2$	$a_1f_1g_1$	$a_2d_2e_1$	$a_2d_1e_2$
$c_1d_2g_1$	$a_1d_2e_2$	$b_1d_2f_2$	$b_1e_1g_1$	$b_2d_1f_2$	$b_1e_2g_2$	$b_2d_2f_1$
$c_2e_2f_2$	$a_2f_2g_1$	$b_2e_2g_1$	$c_1e_2f_1$	$c_2d_2g_2$	$c_2d_1g_1$	$c_1e_1f_2$

The grouping of 15 girls for 7 successive days.

Again, let each of the 15 letters represent correlative pairs

$$p^1p^2,\ a_1^1a_1^2 \ldots g_2^1g_2^2,$$

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then the 35 triads expand into 4×35 or 140 triads of the 30 letters

$$p^1p^2,\ a_1^1a_1^2 \ldots g_2^1g_2^2.$$

Placing a 31st letter q before each of the 15 correlative pairs, we have 155 or 5×31 triads of the 31 letters, and if the 31 letters be replaced by the series

$$abcdefghijklmnopqrstuvwxyz\beta\gamma\delta\theta\phi,$$

we have the 155 triads which may be thus arranged:

ϕab, ϕcd, ϕef, ϕgh, ϕij, ϕkl, ϕmn, ϕop, ϕqr, ϕst, ϕuv, ϕwx, ϕyz, $\phi\beta\gamma$, $\phi\delta\theta$—$ac\delta$, $ad\theta$, $bc\theta$, $bd\delta$—aei, afj, bej, bfi—agk, ahl, bgl, bhk—amq, anr, bmr, bnq—aos, apt, bot, bps—$au\beta$, $av\gamma$, $bw\gamma$, $bv\beta$—axz, awy, bxy, bwz—cek, cfl, del, dfk—cgi, chj, dqj, dhi—cms, cnt, dns, dmt—coq, cpr, dor, dpq—cuy, cvz, duz, dvy—$cx\gamma$, $cw\beta$, $dx\beta$, $dw\gamma$—$eg\delta$, $eh\theta$, $fg\theta$, $fh\delta$—emu, env, fmv, fnu—eow, epx, fox, fpw—$eq\beta$, $er\gamma$, $fq\gamma$, $fr\beta$—esy, etz, fsz, fty—gmw, gnx, hmx, hnw—gou, gpv, hov, hpu—gqy, grz, hqz, hry—$gs\beta$, $gt\gamma$, $hs\gamma$, $ht\beta$—$ik\delta$, $il\theta$, $jk\theta$, $jl\delta$—$im\beta$, $in\gamma$, $jm\gamma$, $jn\beta$—ioy, ipz, joz, jpy—iqu, irv, jqv, jru—isw, itx, jsx, jtw—kmy, knz, lmz, lny—$ko\beta$, $kp\gamma$, $lo\gamma$, $lp\beta$—kqw, krx, lqx, lrw—ksu, ktv, lsv, ltu—$mo\delta$, $mp\theta$, $no\theta$, $np\delta$—$qs\delta$, $qt\theta$, $rs\theta$, $rt\delta$—$uw\delta$, $ux\theta$, $vw\theta$, $vx\delta$—$y\beta\delta$, $y\gamma\theta$, $z\beta\theta$, $z\gamma\delta$.

Let these 15 successive pairs be thus denoted:

$$ab,\ cd,\ ef,\ gh,\ ij,\ kl,\ mn,\ op,\ qr,\ st,\ uv,\ wx,\ yz,\ \beta\gamma,\ \delta\theta,$$

$$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7\ \lambda\ \beta_1\ \beta_2\ \beta_3\ \beta_4\ \beta_5\ \beta_6\ \beta_7,$$

and each of the above quaternions of triads be represented thus:

$ac\delta$, $ad\theta$, $bc\theta$, $bd\delta$ by $\alpha_1\alpha_2\beta_7$, &c.

Hence the 155 triads will be thus written:

$\alpha_1\phi$, $\alpha_2\phi$, $\alpha_3\phi$, $\alpha_4\phi$, $\alpha_5\phi$, $\alpha_6\phi$, $\alpha_7\phi$,

$\lambda\phi$, $\beta_1\phi$, $\beta_2\phi$, $\beta_3\phi$, $\beta_4\phi$, $\beta_5\phi$, $\beta_6\phi$, $\beta_7\phi$,

$\alpha_1\alpha_2\beta_7$—$\alpha_1\alpha_3\alpha_5$—$\alpha_1\alpha_4\alpha_6$—$\alpha_1\alpha_7\beta_1$—$\alpha_1\lambda\beta_2$—$\alpha_1\beta_3\beta_6$—$\alpha_1\beta_4\beta_5$—
$\alpha_2\alpha_3\alpha_6$—$\alpha_2\alpha_4\alpha_5$—$\alpha_2\alpha_7\beta_2$—$\alpha_2\lambda\beta_1$—$\alpha_2\beta_3\beta_5$—$\alpha_2\beta_4\beta_6$—$\alpha_3\alpha_4\beta_7$—
$\alpha_3\alpha_7\beta_3$—$\alpha_3\lambda\beta_4$—$\alpha_3\beta_1\beta_6$—$\alpha_3\beta_2\beta_5$—$\alpha_4\alpha_7\beta_4$—$\alpha_4\lambda\beta_3$—$\alpha_4\beta_1\beta_5$—
$\alpha_4\beta_2\beta_6$—$\alpha_5\alpha_3\beta_7$—$\alpha_5\alpha_7\beta_6$—$\alpha_5\lambda\beta_5$—$\alpha_5\beta_1\beta_3$—$\alpha_5\beta_2\beta_4$—$\alpha_6\alpha_7\beta_5$—
$\alpha_6\lambda\beta_6$—$\alpha_6\beta_1\beta_4$—$\alpha_6\beta_2\beta_3$—$\alpha_7\lambda\beta_7$—$\beta_1\beta_2\beta_7$—$\beta_3\beta_4\beta_7$—$\beta_5\beta_6\beta_7$.

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In each of the above 35 triads of the symbols $\alpha_1\alpha_2 \ldots \lambda \ldots \beta_7$, no two letters occur together twice.

If we select 5 of these triads so as to include each of the 15 symbols such as

$$\alpha_1\alpha_2\beta_7 — \alpha_3\alpha_7\beta_3 — \alpha_4\beta_1\beta_5 — \alpha_5\beta_2\beta_4 — \alpha_6\lambda\beta_6 \ldots\ldots (A),$$

observing that a triad $\alpha_1\alpha_2\beta_7$ is composed of 2 of the 6 letters $a, b, c, d, \delta, \theta$.

Then these 5 triads contain 2 of each of the 30 letters

$$a, b, c, \ldots \delta, \theta.$$

Let each of these letters and ϕ the 31st represent correlative pairs such as a of a^1a^2, &c.

Introducing a 63rd letter μ to these 62 new letters, we get the 31 triads $\mu a^1a^2, \mu b^1b^2 \ldots \mu\phi^1\phi^2$.

If we place under $\mu\phi^1\phi^2$ the 5 triads (A), we have a column of 20 triads of the 30 letters $a, b, \ldots\theta$, each of which occurs twice, no 2 occurring together in different triads.

We have now to assign to the pairs of a's, b's, &c. their values a^1a^2, &c. so that no 2 of the 60 letters $a^1a^2 \ldots \theta^1\theta^2$ occur together in different triads, and so to obtain the 20 triads that are to be placed under $\phi^1\phi^2\mu$, and then to obtain the groups of 20 triads for the other 30 columns.

In order to find the 20 triads for μa^1a^2 and μb^1b^2 since a and b occur twice in the 4 triads

$$ac\delta,\ ad\theta,\ bd\delta,\ bc\theta \text{ of } \alpha_1\alpha_2\beta_7,$$

after writing below μa^1a^2 and μb^1b^2 the 20 triads of the 5 quaternion symbols (A), we must replace the 2 triads under μa^1a^2 containing a, and the 2 under μb^1b^2 containing b by the 2 triads $cd\phi$ and $\delta\theta\phi$.

Thus the 2 columns of 20 triads below μa^1a^2, μb^1b^2 contain $\alpha_1\alpha_2\beta_7$ once, the other 4 triads of (A) twice, and $cd\phi$, $\theta\delta\phi$ twice.

To form the 2 columns under μc^1c^2, μd^1d^2 take a different group similar to (A) containing $\alpha_1\alpha_2\beta_7$. Adopting a similar process we should use $\alpha_1\alpha_2\beta_7$ once, 4 other triads twice, and $ab\phi$, $\theta\delta\phi$ twice.

Similarly for the columns under $\mu\theta^1\theta^2$, $\mu\delta^1\delta^2$ taking another different group similar to (A) containing $\alpha_1\alpha_2\beta_7$, we should use $\alpha_1\alpha_2\beta_7$ a third time, 4 other triads twice, and $ab\phi$, $cd\phi$ twice.

Hence 3 different groups of 5 quaternion triads com-

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mencing with $\alpha_1\alpha_2\beta_7$ furnish the 20 representative triads for the 6 columns under

$$\mu a^1 a^2,\ \mu b^1 b^2,\ \mu c^1 c^2,\ \mu d^1 d^2,\ \mu \delta^1 \delta^2,\ \mu \theta^1 \theta^2.$$

The same process with the other 4 groups of (A) will furnish the columns under the remaining 24 triads.

In order to form these groups we must observe that as each component triad of the quaternion triads, consisting of representative letters resolves itself into 4 triads, each triad of $\alpha_1 \ldots \lambda \ldots \beta_7$ may be used 4 times. Hence we can take the table of triads of these quaternion symbols and mark each triad as we use it that it may not occur more than 4 times.

Then selecting the triads in the order of the letters, we obtain the following sets of 3 columns for each of the triads of (A):

$\alpha_1\alpha_2\beta_7$	$\alpha_1\alpha_2\beta_7$	$\alpha_1\alpha_2\beta_7$	$\alpha_3\alpha_7\beta_3$	$\alpha_3\alpha_7\beta_3$	$\alpha_3\alpha_7\beta_3$
$\alpha_3\beta_1\beta_6$	$\alpha_3\beta_2\beta_5$	$\alpha_3\lambda\beta_4$	$\alpha_4\beta_2\beta_6$	$\alpha_4\alpha_5\alpha_3$	$\alpha_4\alpha_6\alpha_1$
$\alpha_4\alpha_7\beta_4$	$\alpha_4\lambda\beta_3$	$\alpha_4\beta_3\beta_5$	$\alpha_5\alpha_6\beta_7$	$\alpha_6\beta_1\beta_4$	$\alpha_5\lambda\beta_5$
$\alpha_5\lambda\beta_5$	$\alpha_5\alpha_7\beta_6$	$\alpha_5\beta_1\beta_3$	$\lambda\beta_1\alpha_2$	$\beta_5\beta_6\beta_7$	$\beta_1\beta_3\beta_7$
$\alpha_6\beta_2\beta_3$	$\alpha_6\beta_1\beta_4$	$\alpha_6\alpha_7\beta_5$	$\beta_4\beta_5\alpha_1$	$\alpha_1\beta_2\lambda$	$\beta_4\beta_6\alpha_2$
$\alpha_4\beta_1\beta_5$	$\alpha_4\beta_1\beta_5$	$\alpha_4\beta_1\beta_5$	$\alpha_5\beta_2\beta_4$	$\alpha_5\beta_2\beta_4$	$\alpha_5\beta_2\beta_4$
$\alpha_5\alpha_1\alpha_3$	$\alpha_5\alpha_7\beta_6$	$\alpha_5\alpha_6\beta_7$	$\alpha_6\alpha_2\alpha_3$	$\alpha_6\alpha_7\beta_5$	$\alpha_6\alpha_1\alpha_4$
$\alpha_6\beta_2\beta_3$	$\alpha_6\alpha_2\alpha_3$	$\lambda\beta_4\alpha_3$	$\alpha_7\beta_1\alpha_1$	$\lambda\beta_1\alpha_2$	$\alpha_7\lambda\beta_7$
$\alpha_7\lambda\beta_7$	$\lambda\beta_3\alpha_1$	$\beta_3\alpha_2\alpha_7$	$\lambda\beta_3\alpha_4$	$\beta_3\beta_6\alpha_1$	$\beta_1\beta_6\alpha_3$
$\beta_4\beta_6\alpha_2$	$\beta_3\beta_4\beta_7$	$\beta_3\beta_5\alpha_1$	$\beta_5\beta_6\beta_7$	$\beta_7\alpha_3\alpha_4$	$\beta_3\beta_5\alpha_2$

$\alpha_6\lambda\beta_6$	$\alpha_6\lambda\beta_6$	$\alpha_6\lambda\beta_6$
$\alpha_7\beta_1\alpha_1$	$\alpha_7\beta_4\alpha_4$	$\alpha_7\beta_3\alpha_2$
$\beta_2\beta_5\alpha_3$	$\beta_1\beta_2\beta_7$	$\beta_1\beta_5\alpha_5$
$\beta_3\beta_4\beta_7$	$\beta_3\beta_5\alpha_2$	$\beta_4\beta_5\alpha_1$
$\alpha_2\alpha_4\alpha_5$	$\alpha_1\alpha_3\alpha_5$	$\beta_7\alpha_3\alpha_4$

From these 15 columns of triads and the 5 base triads under $\mu\phi^1\phi^2$ the 31 required columns of 21 triads may be formed by applying the same process as that used in the introductory simple problem, for fixing the letters of the 4 representative triads of each of the quaternion triads $\alpha_1\alpha_2\beta_5$, &c.

The triads containing $abcd\delta\theta\phi$ will be determined in the same manner. Thus we are able to write down all the columns which furnish the solution of the problem of 63 girls who are to be so arranged as to walk for 31 successive days in groups of 3, no 2 girls being twice together.

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$\mu a^1 a^2$	$\mu b^1 b^2$	$\mu c^1 c^2$
$b^1 c^1 \theta^2$	$a^1 c^1 \delta^2$	$b^2 d^2 \delta^2$
$b^2 d^1 \delta^1$	$a^2 d^2 \theta^2$	$a^2 d^1 \theta^1$
$c^2 d^2 \phi^2$	$c^2 d^1 \phi^1$	$a^1 b^1 \phi^2$
$\delta^2 \theta^1 \phi^1$	$\delta^1 \theta^1 \phi^2$	$\delta^1 \theta^2 \phi^1$
$e^1 q^1 \beta^2$	$e^1 q^2 \beta^1$	$e^1 s^1 y^2$
$e^2 r^1 \gamma^1$	$e^2 r^2 \gamma^2$	$e^2 t^1 z^1$
$f^2 r^2 \beta^1$	$f^2 r^1 \beta^2$	$f^2 t^2 y^1$
$f^1 q^2 \gamma^2$	$f^1 q^1 \gamma^1$	$f^1 s^2 z^2$
$g^1 m^1 w^2$	$g^1 m^2 w^1$	$g^1 o^1 u^2$
$g^2 n^1 x^1$	$g^2 n^2 x^2$	$g^2 p^1 v^1$
$h^2 n^2 w^1$	$h^2 n^1 w^2$	$h^2 p^2 u^1$
$h^1 m^2 x^2$	$h^1 m^1 x^1$	$h^1 o^2 v^2$
$i^{\cdot 1} o^1 y^2$	$i^{\cdot 1} o^2 y^1$	$i^{\cdot 1} m^1 \beta^2$
$i^{\cdot 2} p^1 z^1$	$i^{\cdot 2} p^2 z^2$	$i^{\cdot 2} n^1 \gamma^1$
$j^{\cdot 2} p^2 y^1$	$j^{\cdot 2} p^1 y^2$	$j^{\cdot 2} n^2 \beta^1$
$j^{\cdot 1} o^2 z^2$	$j^{\cdot 1} o^1 z^1$	$j^{\cdot 1} m^2 \gamma^2$
$k^1 s^1 u^2$	$k^1 s^2 u^1$	$k^1 q^1 w^2$
$k^2 t^1 v^1$	$k^2 t^2 v^2$	$k^2 r^1 x^1$
$l^2 s^2 v^2$	$l^2 s^1 v^1$	$l^2 r^2 w^1$
$l^2 t^2 u^1$	$l^2 t^1 u^2$	$l^2 q^2 x^2$
$\mu d^1 d^2$	$\mu \delta^1 \delta^2$	$\mu \theta^1 \theta^2$
$a^2 c^1 \delta^1$	$a^1 d^2 \theta^1$	$a^1 c^2 \delta^1$
$b^1 c^2 \theta^1$	$b^2 c^2 \theta^2$	$b^1 d^1 \delta^2$
$a^1 b^2 \phi^1$	$a^2 b^1 \phi^1$	$a^2 b^2 \phi^2$
$\delta^2 \theta^2 \phi^2$	$c^1 d^1 \phi^2$	$c^1 d^2 \phi^1$
$e^1 s^2 y^1$	$e^1 o^1 w^2$	$e^1 o^2 w^1$
$e^2 t^1 z^2$	$e^2 p^1 x^1$	$e^2 p^2 x^2$
$f^2 t^1 y^2$	$f^2 p^2 w^1$	$f^2 p^1 w^2$
$f^1 s^1 z^1$	$f^1 o^2 x^2$	$f^1 o^1 x^1$
$g^1 o^2 u^1$	$g^1 s^1 \beta^2$	$g^1 s^2 \beta^1$
$g^2 p^2 v^2$	$g^2 t^1 \gamma^1$	$g^2 t^2 \gamma^2$
$h^2 p^1 u^2$	$h^2 t^2 \beta^1$	$h^2 t^1 \beta^2$
$h^1 o^1 v^1$	$h^1 s^2 \gamma^2$	$h^1 s^1 \gamma^1$
$i^{\cdot 1} m^2 \beta^1$	$i^{\cdot 1} q^1 u^2$	$i^{\cdot 1} q^2 u^1$
$i^{\cdot 2} n^2 \gamma^2$	$i^{\cdot 2} r^1 v^1$	$i^{\cdot 2} r^2 v^2$
$j^{\cdot 2} n^1 \beta^2$	$j^{\cdot 2} r^2 u^1$	$j^{\cdot 2} r^1 w^2$
$j^{\cdot 1} m^1 \gamma^1$	$j^{\cdot 1} q^2 v^2$	$j^{\cdot 1} q^1 v^1$

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$k^1q^2w^1$	$k^1m^1y^2$	$k^1m^2y^1$
$k^2r^2x^2$	$k^2n^1z^1$	$k^2n^2z^2$
$l^2r^1w^2$	$l^2n^2y^1$	$l^2n^1y^2$
$l^1q^1x^1$	$l^1m^2z^2$	$l^1m^1z^1$
μe^1e^2	μf^1f^2	μm^1m^2
$f^2n^1u^1$	$e^1m^1u^2$	$e^2n^1v^1$
$f^1m^1v^2$	$e^2n^2v^2$	$f^2n^2u^2$
$m^2n^2\phi^2$	$m^2n^1\phi^1$	$e^1f^1\phi^2$
$u^2v^1\phi^1$	$u^1v^1\phi^2$	$u^1v^2\phi^1$
$g^2s^2\beta^2$	$g^2s^1\beta^1$	$c^1g^1i^2$
$g^1t^2\gamma^1$	$g^1t^1\gamma^2$	$c^2h^1j^1$
$h^1t^1\beta^1$	$h^1t^2\beta^2$	$d^2h^2i^1$
$h^2s^1\gamma^2$	$h^2s^2\gamma^1$	$d^1g^2j^2$
$i^1k^1\delta^2$	$i^1k^2\delta^1$	$k^2q^2w^2$
$i^2l^1\theta^1$	$i^2l^2\theta^2$	$k^1r^2x^1$
$j^2l^2\delta^1$	$j^2l^1\delta^2$	$l^1r^1w^1$
$j^1k^2\theta^2$	$j^1k^1\theta^1$	$l^2q^1x^2$
$c^1o^1q^2$	$c^1o^2q^1$	$y^1\beta^1\delta^2$
$c^2p^1r^1$	$c^2p^2r^2$	$y^2\gamma^1\theta^1$
$d^2p^2q^1$	$d^2p^1q^2$	$z^2\gamma^2\delta^1$
$d^1o^2r^2$	$d^1o^1r^1$	$z^1\beta^2\theta^2$
$a^2x^1z^1$	$a^2x^2z^2$	$a^1o^1s^2$
$a^1w^1y^2$	$a^1w^2y^1$	$a^2p^1t^1$
$b^2x^2y^1$	$b^2x^1y^2$	$b^2p^2s^1$
$b^1w^2z^2$	$b^1w^1z^1$	$b^1o^2t^2$
μn^1n^2	μu^1u^2	μv^1v^2
$e^2m^1u^1$	$e^1n^2v^1$	$e^1m^2u^1$
$f^1m^2v^1$	$f^2m^2v^2$	$f^1n^1u^2$
$e^1f^2\phi^1$	$e^2f^1\phi^1$	$e^2f^2\phi^2$
$u^2v^2\phi^2$	$m^1n^1\phi^2$	$m^1n^2\phi^1$
$c^1g^2i^1$	$a^1g^1k^2$	$a^1g^2k^1$
$c^2h^2j^2$	$a^2h^1l^1$	$a^2h^2l^2$
$d^2h^1i^2$	$b^2h^2k^1$	$b^2h^1k^2$
$d^1g^1j^1$	$b^1g^2l^2$	$b^1g^1l^1$
$k^2q^1w^1$	$i^2o^2y^2$	$i^2o^1y^1$
$k^1r^1x^2$	$i^1p^2z^1$	$i^1p^1z^2$
$l^1r^2w^2$	$j^1p^1y^1$	$j^1p^2y^2$
$l^2q^2x^1$	$j^2o^1z^2$	$j^2o^2z^1$

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$y^1\beta^2\delta^1$ $q^1s^1\delta^2$ $q^1s^2\delta^1$
$y^2\gamma^2\theta^2$ $q^2t^1\theta^1$ $q^2t^2\theta^2$
$z^2\gamma^1\delta^2$ $r^2t^2\delta^1$ $r^2t^1\delta^2$
$z^1\beta^1\theta^1$ $r^1s^2\theta^2$ $r^1s^1\theta^1$

$a^1o^2s^1$ $c^1w^1\beta^2$ $c^1w^2\beta^1$
$a^2p^2t^2$ $c^2x^1\gamma^1$ $c^2x^2\gamma^2$
$b^2p^1s^2$ $d^2x^2\beta^1$ $d^2x^1\beta^2$
$b^1o^1t^1$ $d^1w^2\gamma^2$ $d^1w^1\gamma^1$

μg^1g^2 μh^1h^2 μq^1q^2

$h^2r^1y^1$ $g^1q^1y^2$ $h^2r^2y^2$
$h^1q^1z^2$ $g^2r^2z^2$ $g^2r^1z^1$
$q^2r^2\phi^2$ $q^2r^1\phi^1$ $g^1h^1\phi^2$
$y^2z^1\phi^1$ $y^1z^1\phi^2$ $y^1z^2\phi^1$

$a^1e^1i'^2$ $a^1e^2i'^1$ $i'^2m^2\beta^2$
$a^2f^1j'^1$ $a^2f^2j'^2$ $i'^1n^2\gamma^1$
$b^1e^2j'^2$ $b^1e^1j'^1$ $j'^1n^1\beta^1$
$b^2f^2i'^1$ $b^2f^1i'^2$ $j'^2m^1\gamma^2$

$k^2s^2u^2$ $k^2s^1u^1$ $c^1e^1k^2$
$k^1t^2v^1$ $k^1t^1v^2$ $c^2f^1l^1$
$l^1t^1u^1$ $l^1t^2u^2$ $d^2f^2k^1$
$l^2s^1v^2$ $l^2s^2v^1$ $d^1e^2l^2$

$m^1o^1\delta^2$ $m^1o^2\delta^1$ $a^2o^2s^2$
$m^2p^1\theta^1$ $m^2p^2\theta^2$ $a^1p^2t^1$
$n^2p^2\delta^1$ $n^2p^1\delta^2$ $b^1p^1s^1$
$n^1o^2\theta^2$ $n^1o^1\theta^1$ $b^2o^1t^2$

$c^2w^2\beta^2$ $c^2w^1\beta^1$ $u^2w^2\delta^2$
$c^1x^2\gamma^1$ $c^1x^1\gamma^2$ $u^1x^2\theta^1$
$d^1x^1\beta^1$ $d^1x^2\beta^2$ $v^2w^1\theta^2$
$d^2w^1\gamma^2$ $d^2w^2\gamma^1$ $v^1x^1\delta^1$

μr^1r^2 μy^1y^2 μz^1z^2

$g^2q^1y^1$ $g^1r^2z^1$ $g^1q^2y^1$
$h^1q^2z^1$ $h^2q^2z^2$ $h^1r^1y^2$
$g^1h^2\phi^1$ $g^2h^1\phi^1$ $g^2h^2\phi^2$
$y^2z^2\phi^2$ $r^1q^1\phi^2$ $q^1r^2\phi^1$

$i'^2m^1\beta^1$ $i'^2k^2\delta^2$ $i'^2k^1\delta^1$
$i'^1n^1\gamma^2$ $i'^1l^2\theta^1$ $i'^1l^1\theta^2$
$j'^1n^2\beta^2$ $j'^1l^1\delta^1$ $j'^1l^2\delta^2$
$j'^2m^2\gamma^1$ $j'^2k^1\theta^2$ $j'^2k^2\theta^1$

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$c^1e^xk^1$	$e^2o^2w^2$	$e^2o^1w^1$
$c^xf^2l^2$	$e^1p^xx^1$	$e^1p^1x^2$
$d^2f^1k^x$	$f^1p^1w^1$	$f^1p^xw^2$
$d^1e^1l^1$	$f^xo^1x^2$	$f^2o^xx^1$
$a^2o^1s^1$	$c^2m^2s^2$	$c^xm^1s^1$
$a^1p^1t^x$	$c^1n^xt^1$	$c^1n^1t^2$
$b^2p^2s^x$	$d^1n^1s^1$	$d^1n^xs^2$
$b^xo^xt^1$	$d^2m^1t^2$	$d^xm^xt^1$
$u^2w^1\delta^1$	$a^1u^1\beta^2$	$a^1u^2\beta^1$
$u^1x^1\theta^2$	$a^2v^1\gamma^1$	$a^xv^x\gamma^2$
$v^1x^x\delta^2$	$b^xv^2\beta^1$	$b^xv^1\beta^2$
$v^2w^x\theta^1$	$b^1u^2\gamma^2$	$b^1u^1\gamma^1$

μi^1i^2	μj^1j^2	μs^1s^2
$j^2t^1w^1$	$i^1s^1w^2$	$j^2t^2w^2$
$j^1s^1x^2$	$i^xt^2x^2$	$i^xt^1x^1$
$s^xt^x\phi^2$	$s^2t^1\phi^1$	$i^1j^1\phi^2$
$w^2x^1\phi^1$	$w^1x^1\phi^x$	$w^1x^2\phi^1$
$c^xe^2k^2$	$c^2e^1k^1$	$k^2m^2y^2$
$c^1f^2l^1$	$c^1f^1l^2$	$k^1n^xz^1$
$d^1f^1k^1$	$d^1f^2k^2$	$l^1n^1y^1$
$d^xe^1l^2$	$d^xe^xl^1$	$l^2m^xz^2$
$a^xm^xq^2$	$a^2m^1q^1$	$c^2o^xq^2$
$a^1n^2r^1$	$a^1n^1r^2$	$c^1p^2r^1$
$b^1n^1q^1$	$b^1n^xq^2$	$d^1p^1q^1$
$b^xm^1r^2$	$b^2m^xr^1$	$d^xo^xr^x$
$g^2o^xu^2$	$g^2o^1u^1$	$a^2u^2\beta^2$
$g^1p^xv^1$	$g^1p^1v^x$	$a^1v^x\gamma^1$
$h^1p^1u^1$	$h^1p^xu^x$	$b^1v^1\beta^1$
$h^2o^1v^2$	$h^2o^xv^1$	$b^xu^1\gamma^x$
$y^2\beta^2\delta^2$	$y^2\beta^1\delta^1$	$e^2g^x\delta^2$
$y^1\gamma^x\theta^1$	$y^1\gamma^1\theta^2$	$e^1h^2\theta^1$
$z^1\gamma^1\delta^1$	$z^1\gamma^x\delta^x$	$f^1h^1\delta^1$
$z^2\beta^1\theta^2$	$z^2\beta^2\theta^1$	$f^xg^1\theta^2$

μt^1t^2	μw^1w^2	μx^1x^2
$j^1s^2x^1$	$i^1t^2x^1$	$i^1s^2w^1$
$i^xs^1w^1$	$j^xs^xx^x$	$j^1t^1w^x$
$i^1j^x\phi^1$	$i^xj^1\phi^1$	$j^xi^x\phi^2$
$w^xx^2\phi^2$	$s^1t^1\phi^2$	$s^1t^2\phi^1$

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$k^2m^1y^1$	$a^1g^1k^2$	$a^1g^2k^1$
$k^1n^1z^2$	$a^2h^1l^1$	$a^2h^2l^2$
$l^1n^2y^2$	$b^2h^2k^1$	$b^2h^1k^2$
$l^2m^2z^1$	$b^1g^2l^2$	$b^1g^1l^1$
$c^2o^1q^1$	$m^2o^2\delta^2$	$m^2o^1\delta^1$
$c^1p^1r^2$	$m^1p^2\theta^1$	$m^1p^1\theta^2$
$d^1p^2q^2$	$n^1p^1\delta^1$	$n^1p^2\delta^2$
$d^2o^2r^1$	$n^2o^1\theta^2$	$n^2o^2\theta^1$
$a^2u^1\beta^1$	$e^2q^2\beta^2$	$e^2q^1\beta^1$
$a^1v^1\gamma^2$	$e^1r^2\gamma^1$	$e^1r^1\gamma^2$
$b^1v^2\beta^2$	$f^1r^1\beta^1$	$f^1r^2\beta^2$
$b^2u^2\gamma^1$	$f^2q^1\gamma^2$	$f^2q^2\gamma^1$
$e^2g^1\delta^1$	$c^1u^1y^2$	$c^1u^2y^1$
$e^1h^1\theta^2$	$c^2v^1z^1$	$c^2v^2z^2$
$f^1h^2\delta^2$	$d^2v^2y^1$	$d^2v^1y^2$
$f^2g^2\theta^1$	$d^1u^2z^2$	$d^1u^1z^1$

μk^1k^2	μl^1l^2	μo^1o^2
$l^2p^1\beta^1$	$k^1o^1\beta^2$	$k^2p^1\gamma^1$
$l^1o^1\gamma^2$	$k^2p^2\gamma^2$	$l^2p^2\beta^2$
$o^2p^2\phi^2$	$o^2p^1\phi^1$	$k^1l^1\phi^2$
$\beta^2\gamma^1\phi^1$	$\beta^1\gamma^1\phi^2$	$\beta^1\gamma^2\phi^1$
$a^1m^1q^2$	$a^1m^2q^1$	$g^2m^2w^2$
$a^2n^1r^1$	$a^2n^2r^2$	$g^1n^2x^1$
$b^2n^2q^1$	$b^2n^1q^2$	$h^1n^1w^1$
$b^1m^2r^2$	$b^1m^1r^1$	$h^2m^1x^2$
$e^1s^1y^2$	$e^1s^2y^1$	$q^2s^2\delta^2$
$e^2t^1z^1$	$e^2t^2z^2$	$q^1t^2\theta^1$
$f^2t^2y^1$	$f^2t^1y^2$	$r^1t^1\delta^1$
$f^1s^2z^2$	$f^1s^1z^1$	$r^2s^1\theta^2$
$u^1w^1\delta^2$	$u^1w^2\delta^1$	$c^2u^2y^2$
$u^2x^1\theta^1$	$u^2x^2\theta^2$	$c^1v^2z^1$
$v^2x^2\delta^1$	$v^2x^1\delta^2$	$d^1v^1y^1$
$v^1w^2\theta^2$	$v^1w^1\theta^1$	$d^2u^1z^2$
$c^1g^1i^2$	$c^1g^2i^1$	$a^2e^2i^2$
$c^2h^1j^1$	$c^2h^2j^2$	$a^1f^2j^1$
$d^2h^2i^1$	$d^2h^1i^2$	$b^1f^1i^1$
$d^1g^2j^2$	$d^1g^1j^1$	$b^2e^1j^2$

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$\mu p^1 p^2$	$\mu\beta^1\beta^2$	$\mu\gamma^1\gamma^2$
$k^2 o^1 \beta^1$	$k^1 p^2 \gamma^1$	$k^1 o^2 \beta^1$
$l^1 o^2 \gamma^1$	$l^2 o^2 \gamma^2$	$l^1 p^1 \beta^2$
$k^1 l^2 \phi^1$	$k^2 l^1 \phi^1$	$k^2 l^2 \phi^2$
$\beta^2 \gamma^2 \phi^2$	$o^1 p^1 \phi^2$	$o^1 p^2 \phi^1$
$g^2 m^1 w^1$	$c^1 m^1 s^2$	$c^1 m^2 s^1$
$g^1 n^1 x^2$	$c^2 n^1 t^1$	$c^2 n^2 t^2$
$h^1 n^2 w^2$	$d^2 n^2 s^1$	$d^2 n^1 s^2$
$h^2 m^2 x^1$	$d^1 m^2 t^2$	$d^1 m^1 t^1$
$q^2 s^1 \delta^1$	$i^2 q^2 u^2$	$i^2 q^1 u^1$
$q^1 t^1 \theta^2$	$i^1 r^2 v^1$	$i^1 r^1 v^2$
$r^1 t^2 \delta^2$	$j^1 r^1 u^1$	$j^1 r^2 u^2$
$r^2 s^2 \theta^1$	$j^2 q^1 v^2$	$j^2 q^2 v^1$
$c^2 u^1 y^1$	$a^2 w^2 y^2$	$a^2 w^1 y^1$
$c^1 v^1 z^2$	$a^1 x^2 z^1$	$a^1 x^1 z^2$
$d^1 v^2 y^2$	$b^1 x^1 y^1$	$b^1 x^2 y^2$
$d^2 u^2 z^1$	$b^2 w^1 z^2$	$b^2 w^2 z^1$
$a^2 e^1 i^1$	$e^1 g^1 \delta^2$	$e^1 g^2 \delta^1$
$a^1 f^1 j^2$	$e^2 h^1 \theta^1$	$e^2 h^2 \theta^2$
$b^1 f^2 i^2$	$f^2 h^2 \delta^1$	$f^2 h^1 \delta^2$
$b^2 e^2 j^1$	$f^1 g^2 \theta^2$	$f^1 g^1 \theta^1$

$\mu\phi\phi^1$	
$a^2 c^2 \delta^2$	$i^2 s^2 w^2$
$a^1 d^1 \theta^2$	$i^1 t^1 x^2$
$b^1 d^2 \delta^1$	$j^1 t^2 w^1$
$b^2 c^1 \theta^1$	$j^2 s^1 x^1$
$e^2 m^2 u^2$	$k^2 o^2 \beta^2$
$e^1 n^1 v^2$	$k^1 p^1 \gamma^2$
$f^1 n^2 u^1$	$l^1 p^2 \beta^1$
$f^2 m^1 v^1$	$l^2 o^1 \gamma^1$
$g^2 q^2 y^2$	
$g^1 r^1 z^2$	
$h^1 r^2 y^1$	
$h^2 q^1 z^1$	

The same process leads to the higher problem of 255 girls for 127 days, the general problem being of $2^{2n} - 1$ girls for $2^{2n-1} - 1$ days.

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MATHEMATICAL NOTES.

By W. H. Besant, M.A.

1. *On a Dynamical Problem.*

Imagine a particle moving in a curve under the action of a force directed to a centre which is coincident with the centre of curvature at an apse; and observe that, when it reaches the apse, the force, mF, is normal, and therefore

$$v^2 = F\rho.$$

The velocity at the apse is therefore the velocity which is requisite for circular motion about the centre of force, and the question naturally arises; "If the particle be projected from the apse with this velocity, what will be its subsequent path?" The following considerations will shew that the path is a circle.

In any central orbit, in which the force is an unique function of the distance, there can be only two apsidal distances, so that if S be the centre of force, and SA, SB, two consecutive aspidal distances, the whole path will consist of a series of arcs, identical with the arc AB, but reversed in position at every apse.

Hence, if O be the centre of curvature at A, and S be supposed in AO produced, and be moved towards O, the arc AB will be diminished, and as S approaches to coincidence with O, will become infinitesimal. The orbit will therefore be the limit of the figure formed by the infinitesimal arcs AB, and this limit is clearly a circle.

If, however, the particle be not projected from the apse, but travel along the curve in a direction CB (fig. 1) towards A, and if, before arriving at A, it reach a point B, such that SB is normal, then the subsequent path will be not the arc AB, but the arc BC, supposed to be turned over, about SB, through two right angles. In this case, when S approaches the point O, the arc AB becomes infinitesimal, and the particle will apparently describe the whole curve, if the point A be a true apse, although in fact it does not pass over the arc AB.

Again, if the point S lie between A and O, the particle will arrive at A, and will pass on, describing the whole curve.

We infer then, that if the centre of force be at the centre of curvature of an apse, the particle, if travelling along the

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curve, will continue to move in the curve; but, if projected from the apse, will move in the circle of curvature.

We may remark in general, that a curve cannot be described by a particle under the action of an attractive central force, which is an assigned function of the distance, unless the centre of force lie in one of the normals dividing the curve symmetrically, and between the curve, and the corresponding vertex of the evolute. If the centre of force lie in any other position, the curve cannot be described, unless it be conceived that the law of force is different on opposite sides of any point at which the direction of motion is perpendicular to the radius vector.

2. *On Aberration.*

The explanations usually given of aberration appear to me to be somewhat incomplete, and I therefore venture on the following remarks.

If we are splashed in the face by rain, while we are either at rest or in motion, we can draw no inference as to the direction of the rain, except that it is somewhere in front of us; and similarly, the mere impact of light on an impressible surface would not of itself give us any information about its direction.

The introduction of a straight tube, which is sometimes employed in the explanation, is hardly satisfactory, seeing that we do not constantly carry straight tubes for the purpose of observing natural phenomena.

We may observe, however, that in the space between the pupil of the eye, and the retina, we have the equivalent of a tube, and the explanation of aberration is then, I think, complete, if we bear in mind that the apparent direction of an object is given by the line joining its image on the retina with the centre of the pupil of the eye.

In (fig. 2), suppose A the centre of the pupil of the eye, and AC the axis of a pencil, which, if the eye be at rest, falls on the retina at C.

Then if the eye be moving in the direction AB, the pencil will impinge on the retina at a point D, and will therefore give the impression of a light travelling in the direction BD.

Hence the angle ADB is the aberration, and

$$\frac{\sin ADB}{\sin ABD} = \frac{AB}{AD} = \frac{\text{velocity of the eye}}{\text{velocity of light}}.$$

Thus, if the motion of the eye be the earth's motion about the sun, ADB is very small, and its circular measure $= \alpha \sin\theta$,

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where θ is the earth's way, and α the ratio of the velocities of the earth and of light.

The question thus treated is simply one of Formal Astronomy, no account being taken of any possible effects on the transit of light, of the motion of the medium through which it is passing.

3. *A formula in Glissettes.*

If a right angle slide on the arc of a curve, $p=f(\phi)$, the motion is equivalent to that of rolling the curve

$$x=f\left(\phi+\frac{\pi}{2}\right)-f'(\phi),\quad y=f(\phi)+f'\left(\phi+\frac{\pi}{2}\right),$$

upon the curve

$$x'=\cos\phi f'\left(\phi+\frac{\pi}{2}\right)+\sin\phi f'(\phi),$$

$$y'=\cos\phi f'(\phi)-\sin\phi f'\left(\phi+\frac{\pi}{2}\right).$$

For (fig. 3) if P and Q be the points of contact, the intersection E, of the normal is the instantaneous centre, and if $TP=x$, $TQ=y$, x and y are the coordinates of the carried locus of E, and since

$$OY=p=f(\phi),$$

and

$$PY=\frac{dp}{d\phi}=f'(\phi),$$

$$TY=f\left(\phi+\frac{\pi}{2}\right)\text{ and }QY'=f'\left(\phi+\frac{\pi}{2}\right).$$

Hence the first two equations follow at once.

For the fixed locus of E let fall the perpendicular EN on the initial line from which ϕ is measured; then if $ON=x'$ and $EN=y'$, the second pair of equations is obtained.

As an example, let the given curve be a three-cusped hypocycloid, so that

$$p=a\cos 3\phi.$$

Then it will be found that

$$x^2+y^2=16a^2\text{ and }x'^2+y'^2=9a^2,$$

so that the motion is produced by rolling a circle of radius $4a$, with internal contact, on a circle of radius $3a$.

This result can be obtained from direct Geometry.

For (fig. 4) if F, G, be two positions of the centre of the rolling circle on a diameter QOQ', the tangents TP, TP' at

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the points P and P' of the curve will be at right angles; and the normals QP, QP' will meet at a point E on the circumference of the fixed circle. It will be easily seen that the line TE passes through O, and is four times the radius (c) of the rolling circle, so that while the fixed locus of E is the circle of radius $3c$, the carried locus is a circle of radius $4c$, and having T for its centre.

In my Notes on Roulettes and Glissettes, I have given for the formulæ in the general case

$$\frac{dx}{d\phi} - y = -\rho, \quad \frac{dy}{d\phi} + x = \rho';$$

it will be seen that these follow at once from the preceding equations.

4. *A property of pedal lines.*

The orthocentre of a triangle, and any point on the circumscribing circle, are equidistant from the pedal line corresponding to that point.

Let S be the point (fig. 5) from which perpendiculars SK, SL are let fall on two of the sides, so that KL is the pedal line. If KS produced meet the circle in V, and AS be joined, the angle

$$VCL = ASK = ALK,$$

and CV is therefore parallel to KL. Also, the point O being the orthocentre, and KL meeting OC in G,

$$\begin{aligned} OG &= OC - VK = CC' - 2.OD - VK \\ &= CC' - VS - 2OD - SK \\ &= 2SK - SK = SK; \end{aligned}$$

hence $OGSK$ is a parallelogram, and therefore O and S are equidistant from GK.

This gives a proof of the known theorem, that *if a parabola touch the sides of a triangle, its directrix passes through the orthocentre.*

For, if S be the focus, which must be on the circumscribing circle, KL is the tangent at the vertex, and therefore O is a point in the directrix.

We may remark that P, the intersection of OS and KL is on the nine-point circle of the triangle.

For the middle point F of OE is the centre of the nine-point circle, and, obviously, FP is half of ES.

April, 1870.

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NOTE ON A BINARY EVECTANT FORM.

By SAMUEL ROBERTS.

1. IF U denote the function $a_0x^n + na_1x^{n-1}y + \ldots + a_ny^n$, the form

$$\left\{a_n\left(\frac{d}{dx_1}\right)^n - na_{n-1}\left(\frac{d}{dx_1}\right)^{n-1}\left(\frac{d}{dy_1}\right) + \ldots\right\}\left(x_1\frac{d}{dx} + y_1\frac{d}{dy}\right)^p U \times \left(x_1\frac{d}{dx} + y_1\frac{d}{dy}\right)^q U \ldots\ldots\ldots\ldots (A),$$

where $p + q = n$, represents a covariant of U of the degree n, and is the first evectant of the invariant

$$\left\{a_n\left(\frac{d}{dx}\right)^n - na_{n-1}\left(\frac{d}{dx}\right)^{n-1}\left(\frac{d}{dy}\right) + \ldots\right\} \times \left\{a_n\left(\frac{d}{dx_1}\right)^n - na_{n-1}\left(\frac{d}{dx_1}\right)^{n-1}\frac{d}{dy_1} + \ldots\right\}\left(x_1\frac{d}{dx} + y_1\frac{d}{dy}\right)^p U \times \left(x_1\frac{d}{dx} + y_1\frac{d}{dy}\right)^q U \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B),$$

as may be shewn in the following manner.

Indicating the evector operator by ∇, and (according to Professor Sylvester's nomenclature) the devector operator $\left\{a_n\left(\frac{d}{dx}\right)^n - \ldots\right\}$, by Δ, we find, if C is the m^{th} evectant of I,

$$\Delta^m C = k\Delta^m\nabla^m I = k'\left(a_n\frac{d}{da_n} + a_{n-1}\frac{d}{da_{n-1}} + \ldots\right)^m I = k''I.$$

Therefore, generally, if C is the m^{th} evectant of I, I is the m^{th} devectant of C.

Consequently, if (A) is a first evectant, it is derived from the form (B). It is to be shewn that the coefficient of x^n in (A) is of the form $\frac{d(B)}{da_n}$ to a numerical factor près.

In order to obtain the x-source of (A), that is to say, the coefficient of x^n therein, we may operate on

$$2\left(a_0x_1^p + pa_1x_1^{p-1}y_1 + \ldots\right)\left(a_0x_1^q + qa_1x_1^{q-1}y_1 + \ldots\right),$$

to which the factor 2 has been added for the sake of symmetry.

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This, again, may be written in the form

$$\begin{aligned}
&(a_p y_1^p + pa_{p-1}y_1^{p-1}x_1 + \ldots + a_0 x_1^p)\, a_0 x_1^q \\
&\qquad + (a_q y_1^q + qa_{q-1}y_1^{q-1}x_1 + \ldots + a_0 x_1^q)\, a_0 x_1^p \\
&+ (a_p y_1^p + \ldots\ldots\ldots\ldots)\, qa_1 x_1^{q-1} y_1 \\
&\qquad + (a_q y_1^q + \ldots\ldots\ldots\ldots)\, pa_1 x_1^{p-1} y_1 \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ (a_p y_1^p + \ldots\ldots\ldots\ldots)\, a_q y_1^q \\
&\qquad + (a_q y_1^q + \ldots\ldots\ldots\ldots)\, a_p y_1^p.
\end{aligned}$$

Instead of actually performing the operations indicated by the devector operator, we may write a_n for x^n, $-a_{n-1}$ for $x_1^{n-1}y$, &c., and to avoid ambiguity of sign, we will suppose that n and q are odd, and consequently, that p is even; the same result will be arrived at on any other supposition relative to this point. Also p, q are > 0. We have then

$$\begin{aligned}
&(a_p a_q - pa_{p-1}a_{q+1} + \ldots + a_0 a_n\ \ )\, a_0 \\
&\qquad + (-\ a_q a_p + qa_{q-1}a_{p+1} - \ldots + a_0 a_n)\, a_0 \\
&+ (-\ a_p a_{q-1} + pa_{p-1}a_q - \ldots - a_0 a_{n-1})\, qa_1 \\
&\qquad + (a_q a_{p-1} - qa_{q-1}a_p + \ldots - a_0 a_{n-1})\, pa_1 \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ (-\ a_p a_0 + pa_{p-1}a_1 - \ldots - a_0 a_p\ \ )\, a_p \\
&\qquad + (-\ a_q a_0 + qa_{q-1}a_1 - \ldots + a_0 a_q\ \ )\, a_p.
\end{aligned}$$

This is the x-source required.

Now take the form

$$\left\{\begin{aligned}
&(a_0 x^p + pa_1 x^{p-1}y + \ldots + a_p y^p)\, x_1^q \\
&\qquad + (a_1 x^p + pa_2 x^{p-1}y + \ldots + a_{p+1} y^p)\, qx_1^{q-1}y_1 + \text{&c.} \\
&\qquad + (a_q x^p + pa_{q+1}x^{p-1}y + \ldots + a_n y^p)\, y_1^q
\end{aligned}\right\}$$

$$\times \left\{\begin{aligned}
&(a_0 x^q + qa_1 x^{q-1}y + \ldots + a_q y^q)\, x_1^p \\
&\qquad + (a_1 x^q + qa_2 x^{q-1}y + \ldots + a_{q+1} y^q)\, px_1^{p-1}y_1 + \text{&c.} \\
&\qquad + (a_p x^q + qa_{q+1}x^{q-1}y + \ldots + a_n y^q)\, y_1^p
\end{aligned}\right\}.$$

Upon this, which represents to a numerical factor *près* $\left(x_1\frac{d}{dx} + y_1\frac{d}{dx}\right)^q U \left(x_1\frac{d}{dx} + y_1\frac{d}{dx}\right)^p U$, we have to operate with the devector symbols relative to x, y; x_1, y_1; but since we shall have to differentiate with regard to a_n, it is only neces-

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sary to collect the terms which may give rise to terms containing a_n. We thus obtain

$$a_n \left\{ \begin{array}{l} (a_q x^p + p a_{q+1} x^{p-1} y + \ldots + a_n y^p)\, y^q y_1^n \\ \qquad + (a_p x^q + q a_{p+1} x^{q-1} y + \ldots + a_n y^q)\, y^p y_1^n \\ + (a_{q-1} x^p + p a_q x^{p-1} y + \ldots + a_{n-1} y^p)\, q y^q y_1^{n-1} x_1 \\ \qquad + (a_{p-1} x^q + q a_p x^{q-1} y + \ldots + a_{n-1} y^p)\, p y^p y_1^{n-1} x_1 \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + (a_0 x^p + p a_1 x^{p-1} y + \ldots + a_p y^p)\, y^q y_1^p x_1^q \\ \qquad + (a_0 x^q + q a_1 x^{q-1} y + \ldots + a_q y^q)\, y^p y_1^q x_1^p \\ \qquad\quad - a_n y^n y_1^n \end{array} \right.$$

$$+ x^n (a_0 x_1^q + q a_1 x_1^{q-1} y_1 + \ldots + a_q y_1^q)(a_0 x_1^p + p a_1 x_1^{p-1} y_1 + \ldots + a_p y_1^p)$$

$$+ x_1^n (a_0 x^q + q a_1 x^{q-1} y + \ldots + a_q y^q)(a_0 x_1^p + p a_1 x^{p-1} y + \ldots + a_q y^p)$$

$$- a_0^2 x^n x_1^n.$$

We have now to write a_n for x_1^n, and x^n, $-a_{n-1}$ for $x_1^{n-1} y_1$ and $x^{n-1} y$, &c., and we obtain

$$a_n \Phi + a_n \Phi - 2 a_0^2 a_n^2,$$

where Φ is the same as the covariant source previously obtained. Calling the invariant, of which the above expression in part (B), we have

$$\frac{d(B)}{da_n} = 2\Phi.$$

Consequently Φ is the source of the first evectant of (B).

If $p = n$, then $q = 0$, and *vice versâ*, and the function operated on is $U.U_1$. If n be odd, the covariant and invariant both vanish. If n be even, the covariant is $U \times$ (quadratic invariant), which is the evectant of (quadratic invariant)^2; so that the foregoing result is true generally.

The invariant (B) is not generally the same for different positive values of p, q consistent with $p + q = n$. In the case of the quintic, however, since the order of the invariant is four, it is unique, viz. J. In the case of the sextic, it is of the form

$$A^2 - \frac{p.p-1.q.q-1}{2} B, \quad (p+q=6),$$

since we have the terms

$$a_0^2 a_6 + \frac{p.p-1.q.q-1}{2.2} a_3^3$$

in the covariant source.

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2. By what precedes a generalization is suggested, which can be most readily arrived at by the aid of the symbolical notation chiefly used by Continental mathematicians.

Let U be equally represented by $(a'x+b'y)^n$, $(a''x+b''y)^n$, &c., $(A'x+B'y)^n$, $(A''x+B''y)^n$, &c., with the understanding that in the formal development a'^n, a''^n, &c., A'^n, A''^n, &c. are to be replaced by a_0; $a'^{n-1}b'$, $a''^{n-1}b''$, &c., $A'^{n-1}B'$, $A''^{n-1}B''$, &c. by a_1 and so on.

Assume now the invariantive form

$$(a'B'-b'A')^{p_1}(a'B''-b'A'')^{p_2}\ldots(a'B^{(k)'}-b'A^{(k)'})^{p_k}$$
$$\times(a''B'-b''A')^{p_2}(a''B''-b''A'')^{p_3}\ldots$$
$$(a''B^{(k-1)'}-b''A^{(k-1)'})^{p_k}(a''B^{(k)'}-b''A^{(k)'})^{p_1}$$
$$\times(a'''B'-b'''A')^{p_3}\ldots$$
$$(a'''B^{(k-2)'}-b''A^{(k-2)'})^{p_k}(a'''B^{(k-1)'}-b''A^{(k-1)'})^{p_1}(a'''B^{(k)'}-b'''A^{(k)'})^{p_2}$$
$$\times\&c. \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (I),$$

where the indices follow the cyclic order

$$\left|\begin{array}{llll} p_1, & p_2, & \ldots\ldots\ldots\ldots & p_k \\ p_2, & p_3, & \ldots\ldots p_k, & p_1 \\ p_3, & \ldots\ldots p_k, & p_1, & p_2 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ p_k, & p_1, & \ldots\ldots p_{k-2}, & p_{k-1} \end{array}\right|,$$

and
$$p_1+p_2+\ldots+p_k=n.$$

In order to obtain the first evectant of (I) we may operate with

$$\Sigma\left(y\frac{d}{da'}-x\frac{d}{db'}\right)^n,$$

where the symbol of summation covers all the quantities a', b'; a'', b''; ... A', B'; A'', B''; ..., the combinations of which are to be ultimately assimilated to a_0, a_1, &c.

The result of the operation is of the form

$$K\left\{\begin{array}{l} \Sigma'\ (A'x+B'y)^{p_1}(A''x+B''y)^{p_2}\ldots(A^{(k)'}x+B^{(k)'}y)^{p_k} \\ \qquad\qquad \times(a''B'-b''A')^{p_2},\ \&c. \\ (-)^{kn}\Sigma''\ (a'x+b'y)^{p_1}(a''x+b''y)^{p_2}\ldots(a^{(k)'}x+b^{(k)'}y)^{p_k} \\ \qquad\qquad \times(A''b'-B''a')^{p_2},\ \&c. \end{array}\right\}$$

the successive forms under the symbols Σ' and Σ'' having the same ultimate significance.

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If k and n are both odd, the evectant vanishes identically and the invariant, from which it is derived, also vanishes. For n even or k even, the evectant does not vanish, and is represented by

$$(A'x+B'y)^{p_1}(A''x+B''y)^{p_2}\ldots(A^{(k)'}x+B^{(k)'}y)^{p_k} \times (a''B'-b''A')^{p_2}, \text{ &c. } \ldots\ldots\ldots\ldots (C),$$

provided that this form does not itself vanish identically.

But the form (C) is equivalent to the result of operating with

$$\left(a''\frac{d}{dy_1}-b''\frac{d}{dx_1}\right)^n\left(a'''\frac{d}{dy_2}-b'''\frac{d}{dx_2}\right)^n\ldots\left(a^{(k)'}\frac{d}{dy_{k-1}}-b^{(k)'}\frac{d}{dx_{k-1}}\right)^n,$$

on the form

$$(A'x+B'y)^{p_1}(A''x+B''y)^{p_2}\ldots(A^{(k)'}x+B^{(k)'}y)^{p_k}(A'x_1+B'y_1)^{p_2}\ldots (A'x_2+B'y_2)^{p_3}\ldots(A'x_{k-1}+B'y_{k-1})^{p_k}\ldots .$$

Representing $\left\{a_0\left(\frac{d}{dy_1}\right)^n-na_1\left(\frac{d}{dy_1}\right)^{n-1}\left(\frac{d}{dx_1}\right)+\ldots\right\}$ by Δ_1, we have, as an equivalent form,

$$\Delta_1\Delta_2\ldots\Delta_{k-1}$$

$$\left\{\begin{array}{l} \left(x_1\frac{d}{dx}+y_1\frac{d}{dy}\right)^{p_1}\left(x_2\frac{d}{dx}+y_2\frac{d}{dy}\right)^{p_2}\ldots\left(x_{k-1}\frac{d}{dx}+y_{k-1}\frac{d}{dy}\right)^{p_{k-1}}U \\ \times\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}\right)^{p_2}\left(x_2\frac{d}{dx}+y_2\frac{d}{dy}\right)^{p_3}\ldots\left(x_{k-1}\frac{d}{dx}+y_{k-1}\frac{d}{dy}\right)^{p_k}U \\ \times\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \times\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}\right)^{p_k}\left(x_2\frac{d}{dx}+y_2\frac{d}{dy}\right)^{p_1}\ldots\left(x_{k-1}\frac{d}{dx}+y_{k-1}\frac{d}{dy}\right)^{p_{k-2}}U \end{array}\right.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (C').$$

Consequently the covariant (C') in which the indices satisfy the cyclic order

$$\begin{array}{l} p_1, p_2, \ldots\ldots\ldots p_k, \\ p_2, p_3, \ldots p_k, \quad p_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ p_k, p_1, \ldots p_{k-2}, p_{k-1}, \end{array}$$

and $\qquad p_1+p_2+\ldots+p_k=n$

is a first evectant when k or n is even. When k is odd and n is odd, the invariant (1) vanishes identically by virtue of opposite signs.

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The preceding result may be regarded as a generalisation of the theorem which we meet at the outset of the theory of forms, that the quadratic invariant of an odd function vanishes and an even function is the evectant of its quadratic invariant. This case corresponds to $k = 1$. The case developed at length in § 1 corresponds of course to $k = 2$.

ELECTRO-DYNAMICS.
AMPERE'S THEORY OF THE SOLENOID.

By Percival Frost, M.A.

In this paper I have endeavoured to obtain some of the formulæ necessary for the calculation of the action of a solenoid in a simpler manner than has been given in any of the treatises with which I am acquainted. The subject itself is so interesting that I may venture to include the fundamental proposition concerning the action of elements of two electric currents upon one another, so as to make the paper as complete as I can without describing fully the experiments by which the law of the mutual action can be established, which experiments may be found in any elementary treatise.

In obtaining the formula for the mutual action of the two elements, I have employed only the experiments which Ampere himself considered rough, in consequence of friction and unavoidable mutual actions of portions of currents neglected in the application of the results of the experiments, but it may be considered that the formula so obtained has passed safely through the severe ordeal of Weber's *Electro-Dynameter*, as well as the severer one of comparison of results of calculations, made upon supposition of its truth, with observation.

1. *Mutual action of elements of two electric currents.*

The mutual action of two elements of currents of different intensity is assumed to vary as the product of the measures of the intensity, and the lengths of the two elements, which product is supposed to represent the product of the quantities

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of electricity which pass through the elements in an unit of time.

The action is also supposed to act in the line joining the elements, and to be some function of the distance of the elements and of their position relative to each other and the joining line.

Experiment proves that the mutual action of two wires is unaltered if any portion of either be removed and replaced by another bent in any manner, so long as no part of it is far removed from the line which was occupied by the removed portion. (Ganot's *Physics*, Art. 725).

This allows us to consider the action of a straight element as the aggregate of the actions of broken elements terminating in its extremities.

Oersted having shewn that the action all about a conducting wire at any section was the same, Liouville deduced from symmetry and the law of equal reason the following facts, (*An. de Ch. et Ph.*, 1829):

(1) That the action of two elements lying in the same straight line takes place in the line joining them, to which experiment adds that it is repulsive or attractive according as the direction of the currents is in the same or opposite direction.

(2) That the action of two parallel elements, each perpendicular to the line joining them, takes place in that line, to which experiment adds that the action is attractive if the direction of the current is the same in both elements, repulsive if in opposite directions.

(3) That there is no action between two elements lying in the same plane, the direction of one of which intersects the other at right angles.

(4) That there is no action between two elements which are each perpendicular to the line joining them and also to each other.

Assume the law of action of two elements at a distance r in the *same* direction, as in case (1), to be $f(r)$, and the law for *parallel* elements, case (2), each perpendicular to their distance to be $\phi(r)$.

Let PQ, $P'Q'$ (fig. 6) be the elements ds, ds' of arcs of two electric currents in which the measures of the intensity are i, i' respectively, $PP' = r$.

QM, $Q'M'$ perpendicular to PP', the line joining them, $Q'N'$ perpendicular to the plane MQP.

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The action of PQ, $P'Q'$ may be replaced by those of the broken lines PMQ and $P'M'N'Q'$.

By (3) and (4) there is no action between PM and $M'N'Q'$, or between QM and $P'M'$, $Q'N'$. The whole action is reduced to those of PM and $P'M'$ and QM and $M'N'$, which, assuming the direction of the currents to be that indicated by the arrows are represented by $ii''.PM.P'M'.F(r)$, and $ii''.QM.M'N'.\phi(r)$, both attractive. Now

$$PM = \frac{dr}{ds}\,ds, \quad P'M' = \frac{dr}{ds'}\,ds'.$$

Also
$$\cos QPM' = -\frac{dr}{ds},$$

$$\cos QPQ' = -\frac{dr}{ds} - \frac{d^2r}{ds'\,ds}\,ds',$$

and $\cos QPQ'$ differs from $\cos QPN'$ by a quantity of the second degree in ds', and

$$QPN' = QPM' - \frac{M'N'}{r};$$

therefore
$$\sin QPM' \cdot \frac{M'N'}{r} = -\frac{d^2r}{ds'\,ds}\,ds';$$

therefore
$$QM.M'N' = -r\,\frac{d^2r}{ds'\,ds}\,ds\,ds'.$$

Hence the whole mutual action (attractive)

$$= ii''\,ds\,ds'\left\{-r\,\frac{d^2r}{ds'\,ds}\,\phi(r) + \frac{dr}{ds}\,\frac{dr}{ds'}\,f(r)\right\}.$$

The mutual action is now fully determined when $\phi(r)$ and $f(r)$ are found.

If ds, ds' be elements of two closed currents, the whole action of these currents is

$$\iint ii''\,ds\,ds'\left\{-r\,\frac{d^2r}{ds'\,ds}\,\phi(r) + \frac{dr}{ds}\,\frac{dr}{ds'}\,f(r)\right\}$$

$$= \int ii''\,ds\left[-r\phi(r)\,\frac{dr}{ds} + \int ds'\,\frac{dr}{ds}\,\frac{d}{ds'}\{r\phi(r)\}\right]$$

$$+ \iint ii''\,ds\,ds'\,\frac{dr}{ds}\,\frac{dr}{ds'}\,f(r),$$

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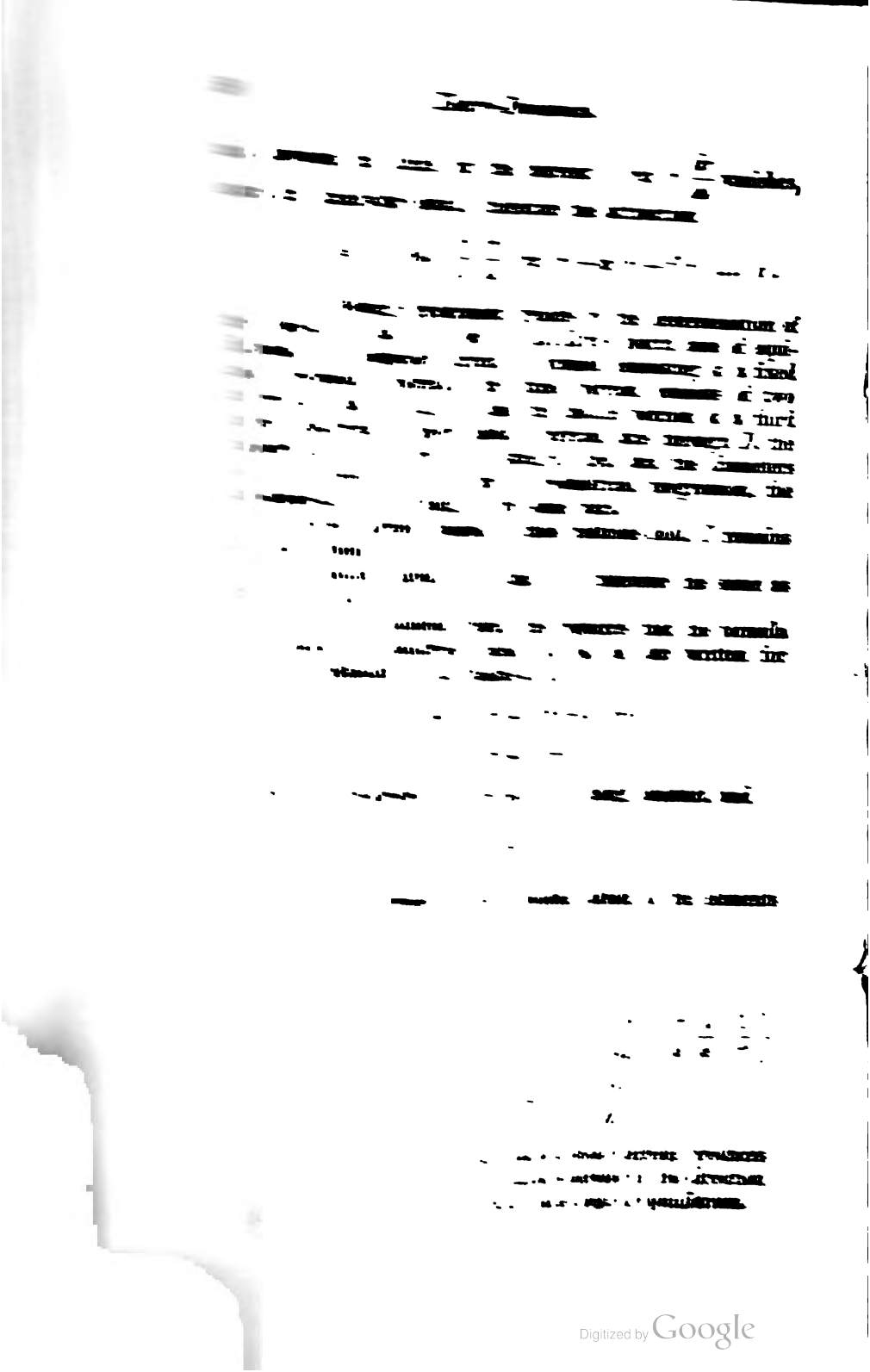

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For the action upon ds estimated parallel to its direction

$$ds \int ds' r^{2k-1} \frac{dr}{ds} \frac{d}{ds'} \left(\frac{1}{r^k} \frac{dr}{ds}\right)$$

$$= ds \int ds' r^{2k-1} \frac{d}{2ds'} \left(\frac{1}{r^k} \frac{dr}{ds}\right)^2,$$

between limits commencing and ending at the same point,

$$= \tfrac{1}{2} ds \left\{ r^{2k-1} \left(\frac{1}{r^k} \frac{dr}{ds}\right)^2 - (2k-1) \int \frac{1}{r^k} \left(\frac{dr}{ds}\right)^2 dr \right\}.$$

The first term vanishes and the second must be made to vanish for all forms of the closed circuit; therefore

$$2k = 1,$$

and the complete expression for the mutual attraction is

$$- \frac{ii' ds ds'}{r^2} \frac{d}{ds'} \left(\frac{1}{r^2} \frac{dr}{ds}\right).$$

2. *To calculate the direction in which the resultant action of a closed current upon an element of a current takes place.*

Since the action of ds' on ds estimated in the direction of ds is

$$- \frac{ii' ds ds'}{r^2} \frac{d}{ds'} \left(\frac{1}{r^2} \frac{dr}{ds}\right) \frac{dr}{ds},$$

the action of the whole current is

$$- \frac{ii' ds}{2} \left\{ \frac{1}{r} \left(\frac{dr}{ds}\right)^2 \right\}$$

between limits which are the same on the two extremities.

The resultant action is therefore perpendicular to the element, this having led to the determination of k.

Take the element at the origin of coordinates and coincident with the axis of x, and let Oz be taken for the direction of the resultant action.

The action parallel to the axis of z

$$= ii' ds \int \frac{ds'}{r^2} \frac{d}{ds'} \left(\frac{1}{r^2} \cdot \frac{x}{r}\right) \frac{z}{r}, \text{ since } \frac{dr}{ds} = -\frac{x}{r},$$

$$= \frac{ii' ds}{2} \int \frac{z}{x} d\left(\frac{x^2}{r^3}\right)$$

$$= -\tfrac{1}{2} ii' ds \int \frac{x^2}{r^3} d\left(\frac{z}{x}\right)$$

$$= \tfrac{1}{2} ii' ds \int \frac{z dx - x dz}{r^3} = ii' ds \int \frac{\alpha}{r^3} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

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where α_y is the area of the projection of $OP'Q'$ on plane zx, and similarly since the action is $Oy = 0$

$$\int \frac{\alpha_y}{r^3} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If α_x', α_y', α_z' correspond to α_x, α_y, α_z for a given system of rectangular axes through the element and l, m, n be the direction cosines of the direction of action with respect to these axes

$$\alpha_y = \alpha_x' l + \alpha_y' m + \alpha_z' n;$$

therefore $$l \int \frac{\alpha_x'}{r^3} + m \int \frac{\alpha_y'}{r^3} + n \int \frac{\alpha_z'}{r^3} = 0, \text{ by (2).}$$

Hence, if a line drawn through the element whose direction cosines, independent of the direction of ds, are proportional to $\int \frac{\alpha_x'}{r^3}$, $\int \frac{\alpha_y'}{r^3}$, $\int \frac{\alpha_z'}{r^3}$ be called the *directrix*, the direction of the resultant is fully defined by being perpendicular to this directrix and to the element.

Referring to (1) since the normal to the area α_y, the elements ds, and the directrix all lie in the plane perpendicular to the direction of the force, it follows that, if ε be the inclination of the element to the directrix, and δ be the projection of $OP'Q'$ on the plane perpendicular to the directrix

$$\alpha_y = \delta \sin \varepsilon.$$

Writing A, B, C for $\int \frac{\alpha_x'}{r^3}$, $\int \frac{\alpha_y'}{r^3}$, $\int \frac{\alpha_z'}{r^3}$, and making

$$D^2 = A^2 + B^2 + C^2,$$

$$\delta = \frac{A\alpha_x' + B\alpha_y' + C\alpha_z'}{D};$$

therefore the magnitude of the resultant is

$$ii'ds \int \frac{\delta}{r^3} \sin \varepsilon$$

$$= ii'ds \int \left(\frac{\alpha_x'}{r^3} \cdot \frac{A}{D} + \frac{\alpha_y'}{r^3} \frac{B}{D} + \frac{\alpha_z'}{r^3} \frac{C}{D} \right) \sin \varepsilon$$

$$= ii'ds D \sin \varepsilon.$$

If ds be referred to any axes, and λ, μ, ν be its direction cosines, the component of the action on ds, parallel to the

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axis of x, is the sum of the actions on $ds\cos\mu$ and $ds\cos\nu$, and these, as by (1), are equivalent to

$$\tfrac{1}{2}ii'\left(ds\cos\mu\int\frac{xdy-ydx}{r^3}+ds\cos\nu\int\frac{xdz-zdx}{r^3}\right)$$

$$=ii'(C\cos\mu-B\cos\nu)\,ds \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

3. *Action of a small plane closed current upon an element of a current.*

Let $lx+my+nz=p$ (fig. 8) be the equation of the plane of the current, and let the current and its projection be cut by two planes parallel to zx at distances y and $y+dy$, P, P' being the intersections of the first plane with the current, Q, Q' with its projection on the plane of xy, and let (x, y, z) and $(x+\delta x, y, z+\delta z)$ denote P and P', we have to obtain

$$2C=\int\frac{xdy-ydx}{r^3},$$

the summation being taken round the curve.

If RR' be the section of the projection by the plane $(y+dy)$, take QR and $Q'R'$ together, the current being in opposite directions in these elements, then

$$\int\frac{xdy}{r^3}=\int\left\{\frac{x}{r^3}-\frac{x+\delta x}{(r+\delta r)^3}\right\}dy$$

$$=\int\left(-\frac{\delta x}{r^3}+\frac{3x\delta r}{r^4}\right)dy,$$

where $r^2=x^2+y^2+z^2$,

therefore $r\delta r=x\delta x+z\delta z$,

and $0=l\delta x+n\delta z$;

therefore $r\delta r=\left(x-\frac{lz}{n}\right)\delta x$,

squares of δx, &c. being neglected; therefore

$$-\frac{\delta x}{r^3}+\frac{3x\delta r}{r^4}=\frac{\delta x}{r^3}\left\{-1+\frac{3x^2}{r^2}-\frac{3lz}{nr^2}\right\};$$

hence $$\int\frac{xdy}{r^3}=\int\frac{\delta x\,dy}{r^3}\left(-1+\frac{3x^2}{r^2}-\frac{3lz}{nr^2}\right).$$

Let λ be the area of the small current

$$\int\delta x\,dy=\lambda n;$$

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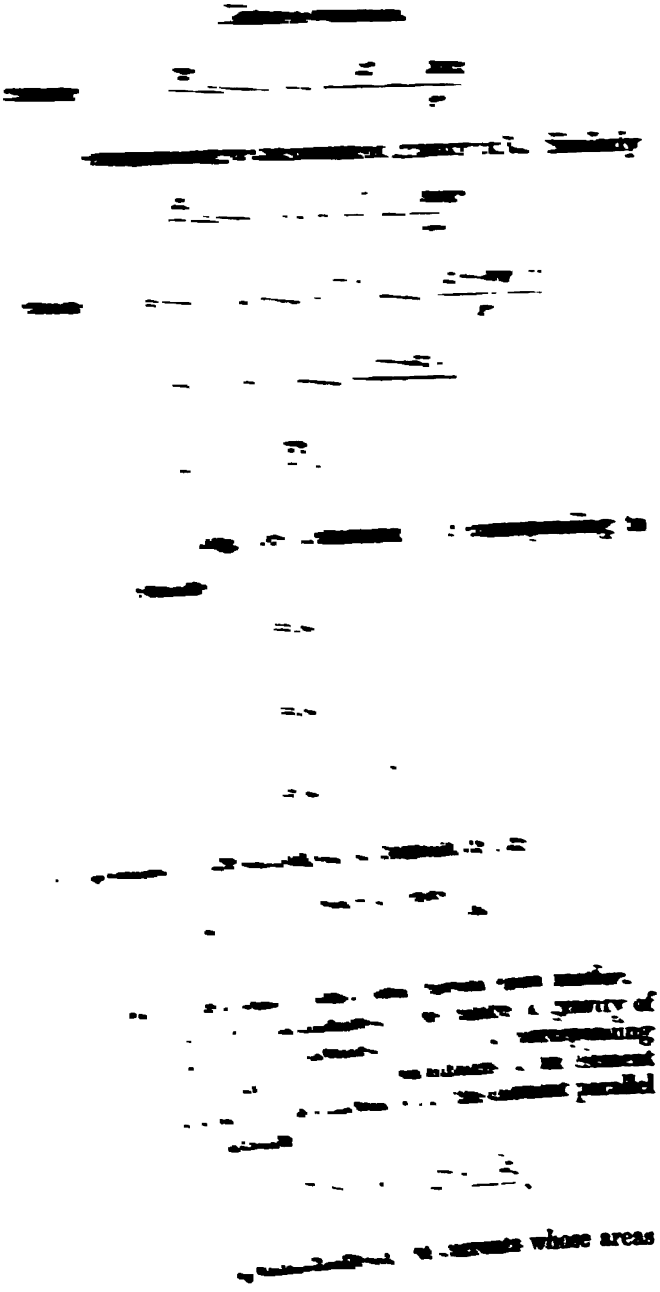

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But if A be found for the directrix in the case of the action of λ' on an element at xyz

$$2A = \int \frac{(y_1 - y)\,dz_1 - (z_1 - z)\,dy_1}{r^3}$$

$$= \lambda' \frac{d}{dp'} \cdot \frac{x' - x}{r^3};$$

therefore $$X = \tfrac{1}{2} ii' \lambda\lambda' \frac{d^2}{dp\,dp'} \cdot \left(\frac{x' - x}{r^3}\right),$$

is the whole action parallel to Ox given by Neumann.

Def. A system of equal small plane closed currents ranged at equal distances, with their centres of gravity on any arbitrary line to which their planes are perpendicular is called by Ampère a *Solenoid.*

5. *Action of a solenoid on a small element of a current.*

Let σ be an arc of the solenoid up to a point xyz, λ the area of a section, $dp = d\sigma$, and if g be the distance of the currents, $\frac{dp}{g}$ is the number of currents in the space dp, the action of the solenoid upon the element at the origin, is obtained from

$$2A = \lambda \int \frac{d\sigma}{g} \frac{d}{d\sigma} \left(\frac{x}{r^3}\right)$$

$$= \frac{\lambda}{g} \left\{ \left(\frac{x}{r^3}\right)_1 - \left(\frac{x}{r^3}\right)_0 \right\},$$

and similar expressions for $2B$ and $2C$, which are independent of the form of the axis of the solenoid between given extremities, denoted by subscripts 1 and 0.

For a solenoid indefinitely extended in one direction

$$2A = \frac{\lambda}{g} \frac{x}{r^3}, \quad 2D = \frac{\lambda}{g} \frac{1}{r^2},$$

and the directrix is the line joining the element with the end of the solenoid, the whole action being

$$\tfrac{1}{2} ii'\,ds \cdot \frac{\lambda}{g} \frac{\sin(r, ds)}{r^2}.$$

The action of a finite solenoid is the resultant of the actions of two solenoids beginning at the two extremities and con-

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tinued each to an infinite distance, the latter action being reversed.

6. *Action of a solenoid on a small plane closed current.*

The action of the element of the solenoid $\lambda d\sigma$ on the current, whose plane area is λ', and p' the perpendicular on it, estimated parallel to the axis of x is

$$\tfrac{1}{2}ii''\lambda'\lambda \frac{d\sigma}{g}\frac{d^2}{dp'd\sigma}\frac{x-x'}{r^3};$$

therefore the whole action of an infinite solenoid is

$$\tfrac{1}{2}ii''\lambda'\lambda \frac{1}{g}\frac{d}{dp'}\frac{x-x_1}{r^3}$$

parallel to Ox.

7. *Action of one solenoid upon another.*

If λ, λ' be the two areas of section; g, g' the distances of the currents, the whole action parallel to Ox is, if both solenoids be infinite,

$$\tfrac{1}{2}ii''\lambda\lambda' \frac{1}{gg'}\frac{x-x'}{r^3}.$$

By considering a finite solenoid as the difference between two infinite solenoids, we obtain the action parallel to Ox

$$\tfrac{1}{2}ii''\lambda\lambda' \frac{1}{gg'}\left\{\frac{x_1-x_1'}{r_1^3}-\frac{x_0-x_1'}{r_0^3}-\frac{x_1-x_0'}{r_1'^3}+\frac{x_0-x_0'}{r_0'^3}\right\},$$

x_1, x_0 and x_1', x_0' corresponding to the extremities of the two solenoids.

The last articles are given more fully by Wiedemann, whose work should be, if possible, read by the student.

I hope on a future occasion to discuss some of the objections which can be justly made to Ampere's Theory of the Mutual Action of Currents.

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ON THE ALGEBRA OF MAGIC SQUARES. No. I.

By Joseph Horner, M.A., Clare College.

If from each term of any arithmetical progression, a, $a+b$, $\ldots a+(r-1)b$ the half sum of the extremes be subtracted, we have the series

$$-\frac{r-1}{2}b,\ -\frac{r-3}{2}b,\ldots\ \frac{r-3}{2}b,\ \frac{r-1}{2}b,$$

which having zero for its sum may be called a zero arithmetical progression. In a similar sense we may speak of a 'zero series,' a 'zero pair,' and so forth.

Dividing each term by b if r be odd, or by $\frac{b}{2}$ if r be even, we have the respective zero arithmetical progressions

$$-\frac{r-1}{2},\ -\frac{r-3}{2}\ldots -1,\ 0,\ 1\ldots\frac{r-3}{2},\ \frac{r-1}{2},$$

$$-\overline{r-1},\ -\overline{r-3},\ldots,\ -3,\ -1,\ 1,\ 3,\ \overline{r-3},\ \overline{r-1}.$$

Inversely, since a and b are any quantities, every arithmetical progression of an odd number of terms may be obtained from the former of these two series by multiplying its terms by a certain quantity, and then adding to each new term another certain quantity. In the like manner, every arithmetical progression of an even number of terms may be formed from the second series.

Assuming then that a magic square is an arithmetical progression of n^2 terms, so arranged in a square figure that the sums of each of the n rows and n columns, and likewise of the two diagonals are the same, and calling *any* square in which this common sum is zero, a zero square, it readily follows from what has been said, that every magic square can be obtained from a corresponding zero magic square formed of one of the two zero arithmetical progressions above, by multiplying the terms by one quantity and then adding another quantity, both of which are readily determined. As to the mechanical operation, this amounts to affirming that, having given a zero magic square, we may substitute for its terms taken in order of algebraical magnitude those of any other

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arithmetical progression taken in order. To zero squares and zero magic squares we may therefore confine the discussion. The sum or difference of the similarly situated elements of two zero squares of equal sides will of course form a zero square of the same side; but its elements will not necessarily constitute an arithmetical progression, or even be all different from each other.

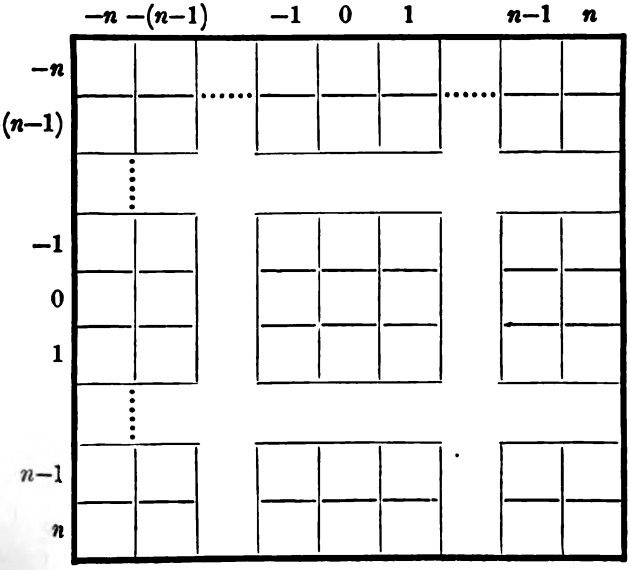
The extremes of a zero arithmetical progression of $(2n+1)^2$ terms are $-2(n^2+n)$ and $2(n^2+n)$, which may be written $-n(2n+1)-n$ and $n(2n+1)+n$. These expressed arithmetically in the scale of notation $(2n+1)$ will be $\bar{n}\,\bar{n}$ and nn, where the negative sign is superposed for convenience. *e.g.* The extremes of a zero arithmetical progression of 5^2 terms are $\bar{2}\,\bar{2}$ and $2\,2$ in the scale of 5. The first $(2n+1)$ terms expressed in the scale of $2n+1$ will be equivalent to the series

$$\bar{n}\,\bar{n},\ \bar{n}(n-1)\ldots\bar{n}\,\bar{1},\ \bar{n}\,0,\ \bar{n}\,1,\ \ldots\bar{n}(n-1),\ \bar{n}\,n,$$

e.g.

$$\bar{2}\,\bar{2},\ \bar{2}\,\bar{1},\ \bar{2}\,0,\ \bar{2}\,1,\ \bar{2}\,2.$$

The next $(2n+1)$ terms will be obtained from these by writing $-(n-1)$ for $-n$, as the first digit of each number, and so on throughout the whole $(2n+1)^2$ terms, as in the table following:



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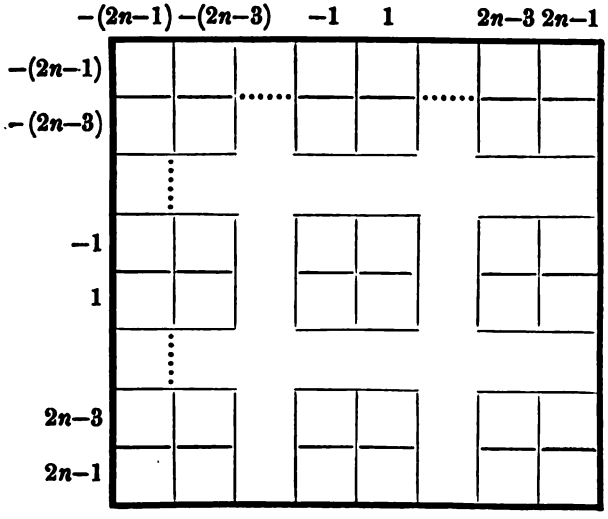
where each compartment is to be filled up with the numbers standing at the head of (1st) the row and (2nd) the column to which it belongs; or *vice versâ*. *e.g.*

	$\bar{2}$	$\bar{1}$	0	1	2
$\bar{2}$	$\bar{2}\,\bar{2}$	$\bar{2}\,\bar{1}$	$\bar{2}\,0$	$\bar{2}\,1$	$\bar{2}\,2$
$\bar{1}$	$\bar{1}\,\bar{2}$	$\bar{1}\,\bar{1}$	$\bar{1}\,0$	$\bar{1}\,1$	$\bar{1}\,2$
0	$0\,\bar{2}$	$0\,\bar{1}$	$0\,0$	$0\,1$	$0\,2$
1	$1\,\bar{2}$	$1\,\bar{1}$	$1\,0$	$1\,1$	$1\,2$
2	$2\,\bar{2}$	$2\,\bar{1}$	$2\,0$	$2\,1$	$2\,2$

In like manner, a zero arithmetical progression of $(2n)^2$ terms has for its extremes $-(4n^2-1)$ and $4n^2-1$, which may be put into the form $-(2n-1)\,2n-(2n-1)$ and $(2n-1)\,2n+(2n-1)$, and therefore expressed arithmetically in the scale of rotation $2n$ by $(\bar{2n}-1)(\bar{2n}-1)$ and $(2n-1)(2n-1)$; and as before the table will be



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Thus, if $n = 3$, the table becomes

	$\bar{5}$	$\bar{3}$	$\bar{1}$	1	3	5
$\bar{5}$	$\bar{5}\,\bar{5}$	$\bar{5}\,\bar{3}$	$\bar{5}\,\bar{1}$	$\bar{5}\,1$	$\bar{5}\,3$	$\bar{5}\,5$
$\bar{3}$	$\bar{3}\,\bar{5}$	$\bar{3}\,\bar{3}$	$\bar{3}\,\bar{1}$	$\bar{3}\,1$	$\bar{3}\,3$	$\bar{3}\,5$
$\bar{1}$	$\bar{1}\,\bar{5}$	$\bar{1}\,\bar{3}$	$\bar{1}\,\bar{1}$	$\bar{1}\,1$	$\bar{1}\,3$	$\bar{1}\,5$
1	$1\,\bar{5}$	$1\,\bar{3}$	$1\,\bar{1}$	$1\,1$	$1\,3$	$1\,5$
3	$3\,\bar{5}$	$3\,\bar{3}$	$3\,\bar{1}$	$3\,1$	$3\,3$	$3\,5$
5	$5\,\bar{5}$	$5\,\bar{3}$	$5\,\bar{1}$	$5\,1$	$5\,3$	$5\,5$

Here we call attention to the fact, that these tables consist of all the permutations with repetitions formed by taking in pairs the elements

$$-n,\ -\overline{n-1},\ \ldots -1,\ 0,\ 1,\ \ldots \overline{n-1},\ n,^{*}$$

and $-\overline{2n-1}, -\overline{2n-3}, \ldots -1, 1, \ldots 2n-3, \overline{2n-1}$, respectively.

It is manifest that each of the tables may be separated into two, the columns of the one being the same as the rows of the other. For instance, that to the side 5 above written consists of the two identical squares

$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$		$\bar{2}$	$\bar{1}$	0	1	2
$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$		$\bar{2}$	$\bar{1}$	0	1	2
0	0	0	0	0	and	$\bar{2}$	$\bar{1}$	0	1	2
1	1	1	1	1		$\bar{2}$	$\bar{1}$	0	1	2
2	2	2	2	2		$\bar{2}$	$\bar{1}$	0	1	2

We may therefore endeavour to form zero squares with the

* It may be shewn as in the text that the extremes of a zero arithmetical progression of $(2n+1)^3$ terms may be written arithmetically $\bar{n}\,\bar{n}\,\bar{n}$ and $n\,n\,n$; and those of a zero arithmetical progression of $(2n)^3$ terms $(2\bar{n}-1)(2\bar{n}-1)(2\bar{n}-1)$ and $(2n-1)(2n-1)(2n-1)$, and that the progressions may be arranged in tables analogous to the above. The same is true generally for zero arithmetical progression of $(2n+1)^r$ and $(2n)^r$ terms.

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odd series $-n, -\overline{n-1}, \ldots \overline{n-1}, n$, which when the similarly situated elements are added together by being placed in juxtaposition in the notational sense, shall exactly restore to us all the permutations above mentioned, and consequently be magic squares. So likewise with the even series $-2n-1$, $-\overline{2n-3}, \ldots 2n-3, 2n-1$. We are not, however, entitled to assert, that no magic squares can be formed except on this principle. The problem of magic cubes, &c. may be treated in the same manner.

Dealing then first with the odd series, let us consider the remainders of the fraction $\frac{ax+by}{N}$, where $N=2n+1$, a and b are given integers, and x, y may take various integer values. We suppose a, b, N to have no common measure; also since if a or b be negative or greater than N, we may evidently replace it by a positive number less than N without affecting the remainder of $\frac{ax+by}{N}$, we assume a and b to be positive and less than N.

Assume $ax+by=mN+r$, where mN is the multiple of N which is *nearest* to $ax+by$, and consequently r admits of any values from $-n$ to n, zero included. Now let $R_N ax+by$ represent r, and suppose that while x remains constant, y takes any N consecutive values, of which any two are represented by y_1 and y_2. Then

$$R_N ax+by_1 - R_N ax+by_2 = R_N b\,(y_1-y_2).$$

First suppose neither a and N, nor b and N have a common measure. Then because y_1-y_2 is numerically less than N, we cannot have $R_N b\,(y_1-y_2)=0$. Consequently as y takes any N consecutive integer values, $R_N ax+by$ receives all the n values of the series $-n, -\overline{n-1}, \ldots 0, \ldots \overline{n-1}, n$, the sum of which is zero. The same is true, if y remains constant and x takes any N consecutive values.

But suppose b and N to have a greatest common measure e, so that $b=eb_1$ and $N=eN_1$. Then $R_N b\,(y_1-y_2)$ becomes $R_N eb_1\,(y_1-y_2)$, which will be zero when $R_{N_1}(y_1-y_2)$ is zero. Now there are e values of y_1-y_2 which are divisible by N_1; hence $R_N ax+by$ will consist of e sets of the same remainders for a given value of x, the sum of which will not necessarily be zero. The same considerations apply to y.

We shall have to consider separately the cases, when N is prime to a and b, and when it has a common measure with one or with each of them separately.

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Taking the former case, put $y =$ any integer value η and give x the N successive values $\xi, \xi+1, \ldots \xi+N-1$, where ξ is any integer. Thus we have the row

$$R_N a\xi + b\eta + R_N a(\xi+1) + b\eta + \ldots R_N(\xi+N-1) + b\eta = 0.$$

Under this arrange corresponding rows by writing for y successively instead of η the terms of the series

$$\eta+1,\ \eta+2, \ldots \eta+N-1.$$

The sum of each of these rows is also zero. So likewise is the sum of each column formed by taking the corresponding terms of each row; *e.g.*

$$R_N a\xi + b\eta + R_N a\xi + b(\eta+1) + \ldots R_N a\xi + b(\eta+N-1) = 0.$$

If then we give to x and y all integer values from ξ to $\xi+N-1$ and from η to $\eta+N-1$ respectively, we form a square, the rows and columns of which are all zero series. The values of x and y at the centre of the square will be $\xi+\frac{N-1}{2}$ and $\eta+\frac{N-1}{2}$; and therefore, proceeding as in coordinate geometry, for the diagonals we have the conditions

$$y-\eta-\frac{N-1}{2} = \pm\left(x-\xi-\frac{N-1}{2}\right),$$

the upper sign applying to the grave (\ ) and the lower to the acute (/) diagonal. Consequently their general terms are

$$R_N(a+b)x - b(\xi-\eta) \quad \text{and} \quad R_N(a-b)x + b(\xi+\eta-1),$$

where x receives all values from ξ to $\xi+N-1$; and by what has preceded the diagonals are, generally speaking, both zero series, and therefore a zero square has been constructed.

If, however, N be a composite number $\zeta\sigma$, a and b must in general be such that neither $a+b$ nor $a-b$ is commensurable with N. For, if $a+b=c\sigma$, then

$$\begin{aligned} &R_N\overline{a+b}(x+\zeta) + b(\eta-\xi) \\ &= R_N\overline{a+b}x + cN + b(\eta-\xi) \\ &= R_N\overline{a+b}x + b(\eta-\xi). \end{aligned}$$

Consequently the grave diagonal consists of a period of ζ terms repeated σ times, which may not be a zero period. And so for the other diagonal.

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e.g. $R_{15}2x+7y$, $\xi=1$, $\eta=1$ has in the acute diagonal the non-zero period $2\ \bar{3}\ 7$; and in the grave the zero period $\bar{6}\ 3\ \bar{3}\ 6\ 0$.

If, however, $R_N a+b=0$, we must have $R_N(\xi-\eta)=0$; or if $a-b=0$, we must have $R_N\xi+\eta-1=0$, and ξ and η must be so assumed as to satisfy these conditions.

It will be noticed that if the values of x be taken in *any* order, the demonstration holds good for the rows, columns, and diagonals; so likewise if the values of y be taken in any order. This, however, cannot be done with *both* x and y. If the order of both be varied it must be in a manner consistent with the conditions for the diagonals.

Examples. If $a=2$, $b=3$, $N=5$, we must have $\eta=\xi$, and the general terms of the diagonals become 0 and $R_5-x+3(2\xi-1)$; giving the two series $0\ 0\ 0\ 0\ 0$ and $2\ 1\ 0\ \bar{1}\ \bar{2}$.

If $a=2$, $b=2$, $N=5$, we must have $\eta=-\xi+1$. Consequently the general terms of the diagonals are $R_5 4x-2(2\xi-1)$ and 0; giving the two series $2\ 1\ 0\ \bar{1}\ \bar{2}$ and $0\ 0\ 0\ 0\ 0$.

The complete squares are

$$\begin{array}{lcl} 0\ 2\ \bar{1}\ 1\ \bar{2} & & 2\ \bar{1}\ 1\ \bar{2}\ 0 \\ \bar{2}\ 0\ 2\ \bar{1}\ 1 & & \bar{1}\ 1\ \bar{2}\ 0\ 2 \\ 1\ \bar{2}\ 0\ 2\ \bar{1} & \text{and} & 1\ \bar{2}\ 0\ 2\ \bar{1} \\ \bar{1}\ 1\ \bar{2}\ 0\ 2 & & \bar{2}\ 0\ 2\ \bar{1}\ 1 \\ 2\ \bar{1}\ 1\ \bar{2}\ 0 & & 0\ 2\ \bar{1}\ 1\ \bar{2} \end{array}$$

There are, however, in general only N different squares obtainable from a given expression $R_N ax+by$. For if the first element of the square be $\mu=R_N a\xi+b\eta$, the rest of the first row may manifestly be derived as follows:

$$\mu_1=R_N\mu+a,\ \mu_2=R_N\mu_1+a,\ \ldots\mu_r=R_N\mu_{r-1}+a,\ \ldots\mu_{N-1}=R_N\mu_{N-2}+a,$$

and the rest of any column headed with μ_r or ν (say),

$$\nu_1=R_N\nu+b,\ \nu_2=R_N\nu_1+b,\ \ldots\nu_r=R_N\nu_{r-1}+b,\ \ldots\nu_{N-1}=R_N\nu_{N-2}+b.$$

Consequently there are as many squares as different values of μ or $R_N ax+by$, which are N in number.

The reader may easily convince himself that no like process is applicable to the formation of even zero squares.

We have next to enquire how the similarly situated

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elements of two of the above zero squares can be combined into magic squares.

Suppose $R_N a_1 x + b_1 y$ to be the general term of a second square, the corresponding values of x and y being the same in both. In order to produce a magic square, or rather a pair of magic squares, from two zero squares, the similarly situated values of $R_N ax + by$ and $R_N a_1 x + b_1 y$ must when placed in juxtaposition produce exactly all the N^2 permutations of N things taken in pairs with repetitions. Now as by the process there will be N^2 pairs formed, it is necessary and sufficient that no two of these pairs be repeated. Suppose then we have

$$R_N ax_1 + by_1 = k \text{ and } R_N a_1 x_1 + b_1 y_1 = l,$$

then we cannot also have both

$$R_N ax_2 + by_2 = k \text{ and } R_N a_1 x_2 + b_1 y_2 = l.$$

That is $$R_N a\,(x_1 - x_2) + b\,(y_1 - y_2) = 0,$$

and $$R_N a_1 (x_1 - x_2) + b_1 (y_1 - y_2) = 0,$$

cannot be simultaneous conditions, whether $x_1 = x_2$, or $y_1 = y_2$, or x_1 and y_1 both differ from x_2 and y_2 respectively.

First, if $x_1 = x_2$, $R_N b\,(y_1 - y_2) = 0$ and $R_N b_1 (y_1 - y_2) = 0$ are incompatible. Now $y_1 - y_2$ is necessarily less than N; consequently if N be a prime, as b and b_1 are by hypothesis less than N, neither of these conditions can exist. But if N be the product of any two factors r and P, then as among the values of $y_1 - y_2$ are included all numbers less than N, and consequently all the factors of N, consider that value of $y_1 - y_2$ which is equal to r; whence it appears that $R_P b = 0$ and $R_P b_1 = 0$ are incompatible, or b and b_1 must not contain in common a factor of N. That is b, b_1 and N must have no common measure. In the same way it can be proved that a, a_1 and N can have no common measure.

Next, suppose that x_1 and x_2, y_1 and y_2 are different. Assume $a\,(x_1 - x_2) + b\,(y_1 - y_2) = Ni$, where i is an integer. Whence

$$R_N aa_1 (x_1 - x_2) + ab_1 (y_1 - y_2) = R_N a_1 Ni + (ab_1 - a_1 b)\,(y_1 - y_2)$$
$$= R_N (ab_1 - a_1 b)\,(y_1 - y_2).$$

Since this remainder cannot be zero, though $y_1 - y_2$ can be any factor of N, another necessary condition is that $ab_1 - a_1 b$ should be a finite integer prime to N.

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Examples 1. $R_N ax+\beta y$ and $R_N ax+\beta y$ cannot form a magic square; for here $ab_1 - a_1 b = a\beta - a\beta = 0$.

Neither can $R\dfrac{ax+\beta y}{a+\beta}$ and $R\dfrac{\beta x+ay}{a+\beta}$; since here

$$R_N ab_1 - a_1 b = R\frac{a^2-\beta^2}{a+\beta} = 0.$$

2. $R_5 2x+3y$ and $R_5 x+3y$ where $x=1, 2 \ldots$ and $y=1, 2 \ldots$ give the magic squares

$0\bar{1}$	20	$\bar{1}1$	12	$\bar{2}\bar{2}$		$\bar{1}0$	02	$1\bar{1}$	21	$\bar{2}\bar{2}$
$\bar{2}2$	$0\bar{2}$	$2\bar{1}$	$\bar{1}0$	11		$2\bar{2}$	$\bar{2}0$	$\bar{1}2$	$0\bar{1}$	11
10	$\bar{2}1$	02	$2\bar{2}$	$\bar{1}\bar{1}$	and	01	$1\bar{2}$	20	$\bar{2}2$	$\bar{1}\bar{1}$
$\bar{1}\bar{2}$	$1\bar{1}$	$\bar{2}0$	01	22		$\bar{2}\bar{1}$	$\bar{1}1$	$0\bar{2}$	10	22
21	$\bar{1}2$	$1\bar{2}$	$\bar{2}\bar{1}$	00		12	$2\bar{1}$	$\bar{2}1$	$\bar{1}\bar{2}$	00.

These expressed in the decimal scale are

$\bar{1}$	10	$\bar{4}$	7	$\overline{12}$		$\bar{5}$	2	4	11	$\overline{12}$
$\bar{8}$	$\bar{2}$	9	$\bar{5}$	6		8	$\overline{10}$	$\bar{3}$	$\bar{1}$	6
5	$\bar{9}$	2	8	$\bar{6}$	and	1	3	10	$\bar{8}$	$\bar{6}$
$\bar{7}$	4	$\overline{10}$	1	12		$\overline{11}$	$\bar{4}$	$\bar{1}$	5	12
11	$\bar{3}$	3	$\overline{11}$	0		7	9	$\bar{9}$	$\bar{7}$	0

Adding 13 to each term we clear negative numbers; thus

12	23	9	20	1		8	15	17	24	1
5	11	22	8	19		21	3	10	12	19
18	4	15	21	7	and	14	16	23	5	7
6	17	3	14	25		2	9	12	18	25
24	10	16	2	13		22	24	4	6	13

in each of which the common sum is 65.

Everton Vicarage, near St. Neots.

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NOTE TO THE ARTICLE ON PORISMS, QUARTERLY JOURNAL, No. 30.

By Joseph Wolstenholme, M.A.

THE equations

$$\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$
$$=\cot\frac{\beta}{2}\cot\frac{\gamma}{2}+\cot\frac{\gamma}{2}\cot\frac{\alpha}{2}+\cot\frac{\alpha}{2}\cot\frac{\beta}{2}=k,$$

are equivalent to the system

$$\left(k-\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\right)\left(k-\cot\frac{\beta}{2}\cot\frac{\gamma}{2}\right)$$
$$=\left(\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}\right)\left(\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}\right),\ \&c.,$$

that is, to the system

$$2(1-k)\cos\beta\cos\gamma+(k^2-1)\sin\beta\sin\gamma=2(1+k),\ \&c...(A).$$

But, if the equations of two concentric conics referred to their common system of conjugate diameters be

$$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,\quad S'\equiv\frac{x^2}{a'^2}+\frac{y^2}{b'^2}-1=0,$$

and if a triangle circumscribe S, the points of contact being $(a\cos\alpha,\ b\sin\alpha)$, $(a\cos\beta,\ b\sin\beta)$, and $(a\cos\gamma,\ b\sin\gamma)$, the conditions that its angular points may lie on S' are

$$\left(-\frac{a^2}{a'^2}+\frac{b^2}{b'^2}+1\right)\cos\beta\cos\gamma+\left(\frac{a^2}{a'^2}-\frac{b^2}{b'^2}+1\right)\sin\beta\sin\gamma$$
$$=\frac{a^2}{a'^2}+\frac{b^2}{b'^2}-1,\ \&c.\ \ldots\ldots\ldots\ldots\ (B).$$

The systems (A), (B) will coincide if

$$\frac{\dfrac{a^2}{a'^2}}{(1+k)^2}=\frac{\dfrac{b^2}{b'^2}}{4}=\frac{1}{(1-k)^2},$$

which lead at once to the condition $\dfrac{a}{a'}\pm\dfrac{b}{b'}\pm1=0$.

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Now if Δ be the area of the triangle formed by joining the points of contact, and Δ' that of the circumscribed triangle,

$$\frac{\Delta}{\Delta'} = \pm 2 \cos\frac{\beta-\gamma}{2} \cos\frac{\gamma-\alpha}{2} \cos\frac{\alpha-\beta}{2}$$

$$= \pm 2 \frac{(1+mn)(1+nl)(1+lm)}{(1+l^2)(1+m^2)(1+n^2)};$$

denoting $\tan\frac{\alpha}{2}$, $\tan\frac{\beta}{2}$, $\tan\frac{\gamma}{2}$ by l, m, n.

But, subject to the conditions

$$mn+nl+lm = \frac{1}{mn} + \frac{1}{nl} + \frac{1}{lm} = k,$$

$$(1+mn)(1+nl)(1+lm) = (1+k)(1+l^2m^2n^2),$$

$$(1+l^2)(1+m^2)(1+n^2) = (1-k)^2(1+l^2m^2n^2),$$

whence $$\frac{\Delta}{\Delta'} = \pm \frac{2(1+k)}{(1-k)^2}, \text{ or } \frac{\Delta^2}{\Delta'^2} = \frac{4(1+k)^2}{(1-k)^4} = \frac{a^2b^2}{a'^2b'^2}.$$

It follows that if two concentric conics be such that triangles can be inscribed in one and circumscribed to the other, and ABC be any such triangle, A', B', C' the points of contact of the sides, then $\Delta ABC : \Delta A'B'C'$ is a constant ratio, which is also the ratio of the areas of the conics when they are ellipses, and is the ratio of the areas of the triangles intercepted between any tangent and the asymptotes, when they are hyperbolas.

We also see that the system of equations

$$\frac{\cos\beta\cos\gamma}{A} + \frac{\sin\beta\sin\gamma}{B} + \frac{1}{A+B} = 0, \text{ \&c.},$$

is equivalent to either of the systems

(1) $$\tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} + \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

$$= \cot\frac{\beta}{2}\cot\frac{\gamma}{2} + \cot\frac{\gamma}{2}\cot\frac{\alpha}{2} + \cot\frac{\alpha}{2}\cot\frac{\beta}{2} = -\left(1+\frac{2A}{B}\right),$$

(2) $$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0,$$

$$2\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2} = -\frac{AB}{(A+B)^2}.$$

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ON THE STRESS EXERTED BY A RIGID BODY ON A FIXED POINT RIGIDLY CONNECTED WITH THE BODY, WHICH IS REVOLVING SPONTANEOUSLY ABOUT THE POINT.

By WILLIAM WALTON, M.A., Fellow of Trinity Hall.

THE object of this article is to prove that the square of the stress on the fixed point is, throughout the motion, equal to $u+v\left(\frac{\omega}{\rho}\right)^2$, where u and v are certain constants, ω the angular velocity of the body about the instantaneous axis, and ρ the portion of the instantaneous axis which is intercepted between the fixed point and a certain centric quadric surface, fixed in the body, of which the fixed point is the centre.

Let the body be referred to a system of fixed rectangular axes passing through the fixed point. Let ω be the angular velocity of the body about the instantaneous axis at any time t, and let ω_x, ω_y, ω_z, be the components of the angular velocity about the axes of coordinates.

Let M denote the mass of the body, X, Y, Z, the components of the reaction of the fixed point, and α, β, γ, the coordinates of the centre of gravity of the body. Then

$$X=M\frac{d^2\alpha}{dt^2},\quad Y=M\frac{d^2\beta}{dt^2},\quad Z=M\frac{d^2\gamma}{dt^2}.$$

Now $\frac{d\alpha}{dt}=\omega_y\gamma-\omega_z\beta$, $\frac{d\beta}{dt}=\omega_z\alpha-\omega_x\gamma$, $\frac{d\gamma}{dt}=\omega_x\beta-\omega_y\alpha$;

hence $\frac{d^2\alpha}{dt^2}=\gamma\frac{d\omega_y}{dt}-\beta\frac{d\omega_z}{dt}+\omega_y(\omega_x\beta-\omega_y\alpha)-\omega_z(\omega_z\alpha-\omega_x\gamma)$,

with analogous expressions for $\frac{d^2\beta}{dt^2}$, $\frac{d^2\gamma}{dt^2}$.

Imagine the axes of coordinates to coincide with the principal axes of the body at the end of the time t; and let ω_1, ω_2, ω_3, be the corresponding values of ω_x, ω_y, ω_z: then, as

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is well known, $\frac{d\omega_x}{dt}$, $\frac{d\omega_y}{dt}$, $\frac{d\omega_z}{dt}$, are respectively equal to $\frac{d\omega_1}{dt}$, $\frac{d\omega_2}{dt}$, $\frac{d\omega_3}{dt}$. Thus, α, β, γ, now denoting the coordinates of the centre of gravity in relation to the principal axes as axes of coordinates,

$$\frac{d^2\alpha}{dt^2} = \gamma\frac{d\omega_2}{dt} - \beta\frac{d\omega_3}{dt} + \omega_2(\omega_1\beta - \omega_2\alpha) - \omega_3(\omega_3\alpha - \omega_1\gamma),$$

with analogous expressions for $\frac{d^2\beta}{dt^2}$ and $\frac{d^2\gamma}{dt^2}$.

Let a, b, c, be the moments of inertia about the principal axes through the fixed point: then, by Euler's formulæ,

$$a\frac{d\omega_1}{dt} = (b-c)\,\omega_2\omega_3, \quad b\frac{d\omega_2}{dt} = (c-a)\,\omega_3\omega_1, \quad c\frac{d\omega_3}{dt} = (a-b)\,\omega_1\omega_2.$$

Hence $$\frac{X}{M} = (b+c-a).\frac{\omega_1}{bc}.(b\beta\omega_2 + c\gamma\omega_3) - \alpha(\omega_2^2 + \omega_3^2),$$

$$\frac{Y}{M} = (c+a-b).\frac{\omega_2}{ca}.(c\gamma\omega_3 + a\alpha\omega_1) - \beta(\omega_3^2 + \omega_1^2),$$

$$\frac{Z}{M} = (a+b-c).\frac{\omega_3}{ab}.(a\alpha\omega_1 + b\beta\omega_2) - \gamma(\omega_1^2 + \omega_2^2).$$

Let R be the resultant pressure on the fixed point: then

$$\frac{R^2}{M^2} = \left\{\frac{b+c-a}{bc}.\omega_1.(b\beta\omega_2 + c\gamma\omega_3) - \alpha(\omega_2^2 + \omega_3^2)\right\}^2$$
$$+ \left\{\frac{c+a-b}{ca}.\omega_2.(c\gamma\omega_3 + a\alpha\omega_1) - \beta(\omega_3^2 + \omega_1^2)\right\}^2$$
$$+ \left\{\frac{a+b-c}{ab}.\omega_3.(a\alpha\omega_1 + b\beta\omega_2) - \gamma(\omega_1^2 + \omega_2^2)\right\}^2.$$

In the expression for $\frac{R^2}{M^2}$ the coefficient of α^2 is equal to

$$(\omega_2^2 + \omega_3^2)^2 + \frac{\omega_1^2\omega_2^2}{c^2}(c+a-b)^2 + \frac{\omega_1^2\omega_3^2}{b^2}(a+b-c)^2,$$

or, since $\omega_2^2 + \omega_3^2 = \omega^2 - \omega_1^2$, to

$$\omega^4 + \omega_1^2\left\{\omega_1^2 + \frac{(c+a-b)^2}{c^2}\omega_2^2 + \frac{(a+b-c)^2}{b^2}\omega_3^2 - 2\omega^2\right\}.$$

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But, as is well known,

$$\omega_1^2 = \frac{bc}{(c-a)(a-b)}(\lambda_1 - \omega^2),$$

$$\omega_2^2 = \frac{ca}{(a-b)(b-c)}(\lambda_2 - \omega^2),$$

$$\omega_3^2 = \frac{ab}{(b-c)(c-a)}(\lambda_3 - \omega^2),$$

where

$$\lambda_1 = \frac{f(b+c) - g^2}{bc},$$

$$\lambda_2 = \frac{f(c+a) - g^2}{ca},$$

$$\lambda_3 = \frac{f(a+b) - g^2}{ab}.$$

Thus $\omega_1^2 + \frac{(c+a-b)^2}{c^2}\omega_2^2 + \frac{(a+b-c)^2}{b^2}\omega_3^2 - 2\omega^2$

$$= -\omega^2\left\{\frac{bc}{(c-a)(a-b)} + \frac{a(c+a-b)^2}{c(a-b)(b-c)} + \frac{a(a+b-c)^2}{b(b-c)(c-a)} + 2\right\}$$

$$+ \frac{\lambda_1 bc}{(c-a)(a-b)} + \frac{\lambda_2 a(c+a-b)^2}{c(a-b)(b-c)} + \frac{\lambda_3 a(a+b-c)^2}{b(b-c)(c-a)}.$$

But, as may be ascertained by the performance of obvious algebraic operations, the coefficient of $-\omega^2$ in this last expression is equal to

$$\frac{(a-b)(a-c)}{bc}.$$

Hence the coefficient of α^2 in the expression for $\frac{R^2}{M^2}$ is equal to

$$\omega^4 + \omega_1^2\left\{\frac{\lambda_1 bc}{(c-a)(a-b)} + \frac{\lambda_2 a(c+a-b)^2}{c(a-b)(b-c)}\right.$$

$$\left. + \frac{\lambda_3 a(a+b-c)^2}{b(b-c)(c-a)} - \frac{(a-b)(a-c)}{bc}\omega^2\right\}$$

$$= \omega_1^2\left\{\frac{\lambda_1 bc}{(c-a)(a-b)} + \frac{\lambda_2 a(c+a-b)^2}{c(a-b)(b-c)} + \frac{\lambda_3 a(a+b-c)^2}{b(b-c)(c-a)}\right\} + \lambda_1\omega^2$$

$$= \frac{\omega_1^2}{bc(b-c)(c-a)(a-b)}\{b^2c^2(b-c)\lambda_1 + ab(c-a)(c+a-b)^2\lambda_2$$

$$+ ac(a-b)(a+b-c)^2\lambda_3\} + \lambda_1\omega^2.$$

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Now, substituting for $\lambda_1, \lambda_2, \lambda_3$, their values, we easily see that

$b^2c^2(b-c)\lambda_1 + ab(c-a)(c+a-b)^2\lambda_2 + ac(a-b)(a+b-c)^2\lambda_3$

is equal to

$$\frac{f}{bc}\{b^2c^2(b^2-c^2) + b^2(c^2-a^2)(c+a-b)^2 + c^2(a^2-b^2)(a+b-c)^2\}$$

$$-\frac{g^2}{bc}\{b^2c^2(b-c) + b^2(c-a)(c+a-b)^2 + c^2(a-b)(a+b-c)^2\}.$$

The coefficients of $\frac{f}{bc}$ and $-\frac{g^2}{bc}$ in this expression may, by the performance of obvious algebraic operations, be proved to be equal respectively to

$$(b-c)(c-a)(a-b)\{a^2(b+c) - a(b-c)^2 - bc(b+c)\},$$

and $(b-c)(c-a)(a-b)\{a(b+c) - (b^2+c^2-bc)\}$.

Hence the coefficient of α^2 in the expression for $\frac{R^2}{M^2}$ is equal to

$$\frac{\omega_1^2}{b^2c^2}[f\{a^2(b+c) - a(b-c)^2 - bc(b+c)\}$$
$$- g^2\{a(b+c) - (b^2+c^2-bc)\}] + \frac{f(b+c)-g^2}{bc}.\omega^2.$$

But $\frac{f(b+c)-g^2}{bc}.\omega^2 = \lambda_1^2 - \frac{(c-a)(a-b)}{b^2c^2}\{f(b+c)-g^2\}\omega_1^2$;

hence the coefficient of α^2 is equal to

$$\lambda_1^2 + \frac{\omega_1^2}{b^2c^2}[f\{a^2(b+c) - a(b-c)^2 - bc(b+c)\}$$
$$- g^2\{a(b+c) - (b^2+c^2-bc)\}$$
$$- (c-a)(a-b)\{f(b+c)-g^2\}]$$
$$= \left\{\frac{f(b+c)-g^2}{bc}\right\}^2 + \frac{\omega_1^2}{b^2c^2}.\{2fa(ab+ac-b^2-c^2)$$
$$- g^2(a+b-c)(a+c-b)\}.$$

Again, the coefficient of $2\beta\gamma$ in the expression for $\frac{R^2}{M^2}$ is equal to

$$\frac{\omega_2\omega_3}{abc}\{a(b+c-a)^2\omega_1^2 - bc(c+a-b)(\omega_3^2+\omega_1^2)$$
$$- bc(a+b-c)(\omega_1^2+\omega_2^2)\}$$

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$$= \frac{\omega_2\omega_3}{abc}\{a(b+c-a)^2\omega_1^2 + bc(c+a-b)\omega_2^2 + bc(a+b-c)\omega_3^2 - 2abc\omega^2\}$$

$$= \omega_2\omega_3\left\{\frac{(b+c-a)^2}{(c-a)(a-b)}(\lambda_1-\omega^2) + \frac{c(c+a-b)}{(a-b)(b-c)}(\lambda_2-\omega^2) + \frac{b(a+b-c)}{(b-c)(c-a)}(\lambda_3-\omega^2) - 2\omega^2\right\}$$

$$= \frac{\omega_2\omega_3}{(b-c)(c-a)(a-b)}\{(b-c)(b+c-a)^2(\lambda_1-\omega^2) + c(c-a)(c+a-b)(\lambda_2-\omega^2) + b(a-b)(a+b-c)(\lambda_3-\omega^2) - 2(b-c)(c-a)(a-b)\omega^2\}.$$

But the expression

$$(b-c)(b+c-a)^2 + c(c-a)(c+a-b) + b(a-b)(a+b-c) + 2(b-c)(c-a)(a-b)$$

may be shewn, by the performance of obvious algebraic operations, to be equal to zero.

Consequently the coefficient of $2\beta\gamma$ is equal to

$$\frac{\omega_2\omega_3}{(b-c)(c-a)(a-b)}\{(b-c)(b+c-a)^2\lambda_1 + c(c-a)(c+a-b)\lambda_2 + b(a-b)(a+b-c)\lambda_3\}$$

$$= \frac{\omega_2\omega_3}{abc(b-c)(c-a)(a-b)}[f\{a(b^2-c^2)(b+c-a)^2 + bc(c^2-a^2)(c+a-b) + bc(a^2-b^2)(a+b-c)\} - g^2\{a(b-c)(b+c-a)^2 + bc(c-a)(c+a-b) + bc(a-b)(a+b-c)\}].$$

Now, by obvious operations, we may shew that the coefficient of f in the brackets [ ] is equal to

$$(b-c)(c-a)(a-b)\{b^2+c^2-a(b+c)\},$$

and that the coefficient of $-g^2$ within the same brackets is equal to

$$(b-c)(c-a)(a-b)(b+c-a).$$

Hence the coefficient of $2\beta\gamma$ is equal to

$$\frac{\omega_2\omega_3}{abc}\{f(b^2+c^2-ab-ac) - g^2(b+c-a)\}.$$

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Thus, finally, in virtue of symmetry, we see that

$$\frac{R^2}{M^2} = \alpha^2 \left\{\frac{f(b+c)-g^2}{bc}\right\}^2 + \beta^2 \left\{\frac{f(c+a)-g^2}{ca}\right\}^2 + \gamma^2 \left\{\frac{f(a+b)-g^2}{ab}\right\}^2$$

$$+ \frac{\alpha^2\omega_1^2}{b^2c^2} \{2af(ab+ac-b^2-c^2) - g^2(a+b-c)(a+c-b)\}$$

$$+ \frac{\beta^2\omega_2^2}{c^2a^2} \{2bf(bc+ba-c^2-a^2) - g^2(b+c-a)(b+a-c)\}$$

$$+ \frac{\gamma^2\omega_3^2}{a^2b^2} \{2cf(ca+cb-a^2-b^2) - g^2(c+a-b)(c+b-a)\}$$

$$+ \frac{2\beta\gamma\omega_2\omega_3}{abc} \{f(b^2+c^2-ab-ac) - g^2(b+c-a)\}$$

$$+ \frac{2\gamma\alpha\omega_3\omega_1}{abc} \{f(c^2+a^2-bc-ba) - g^2(c+a-b)\}$$

$$+ \frac{2\alpha\beta\omega_1\omega_2}{abc} \{f(a^2+b^2-ca-cb) - g^2(a+b-c)\}.$$

The expression for $\frac{R^2}{M^2}$, which originally presented itself in a biquadratic form in ω_1, ω_2, ω_3, has been therefore finally transformed into an equivalent quadratic form.

Suppose that $b=a$: then, ω being supposed to be finite, the equation

$$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$$

shews that each of the quantities ω_1, ω_2, ω_3, must also be finite. Therefore either of the equations

$$\omega_1^2 = \frac{bc}{(c-a)(a-b)} \left\{\frac{f(b+c)-g^2}{bc} - \omega^2\right\},$$

$$\omega_2^2 = \frac{ca}{(a-b)(b-c)} \left\{\frac{f(c+a)-g^2}{ca} - \omega^2\right\},$$

shews that $$\omega^2 = \frac{f}{a} + \frac{af-g^2}{ca}.$$

Hence $$\omega_3^2 = \frac{ab}{(b-c)(c-a)} \left\{\frac{f(a+b)-g^2}{ab} - \omega^2\right\}$$

$$= \frac{af-g^2}{c(a-c)},$$

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and therefore $\omega_1^2 + \omega_2^2 = \omega^2 - \omega_3^2$

$$= \frac{f}{a} + \frac{af - g^2}{ac} - \frac{af - g^2}{c(a-c)}$$

$$= \frac{f}{a} - \frac{af - g^2}{a(a-c)}.$$

Then $$\frac{R^2}{M^2} = (\alpha^2 + \beta^2)\left\{\frac{f(a+c) - g^2}{ac}\right\}^2 + \gamma^2\left\{\frac{2fa - g^2}{a^2}\right\}^2$$

$$+ \frac{\alpha^2\omega_1^2 + \beta^2\omega_2^2}{a^2c}\{2a(a-c)f - g^2(2a-c)\}$$

$$+ \frac{c\gamma^2\omega_3^2}{a^4}\{4af(c-a) - g^2c\}$$

$$+ \frac{2\beta\gamma\omega_2\omega_3 + 2\gamma\alpha\omega_3\omega_1}{a^2}\{f(c-a) - g^2\}$$

$$+ \frac{2\alpha\beta\omega_1\omega_2}{a^2}\left\{2a\,\frac{af - g^2}{c} + g^2 - 2af\right\}.$$

Next suppose that $c = 0$: then, since ω^2 is finite, the equation

$$\omega^2 = \frac{f}{a} + \frac{af - g^2}{ca}$$

shews that $\frac{af - g^2}{c}$ is finite, and therefore that $af = g^2$. Hence

$$\frac{R^2}{M^2} = (\alpha^2 + \beta^2)\left(\frac{fa - g^2}{ac} + \frac{f}{a}\right)^2 + \gamma^2\left(\frac{f}{a}\right)^2$$

$$+ \frac{1}{a^2}(\alpha^2\omega_1^2 + \beta^2\omega_2^2)\left(2a \cdot \frac{af - g^2}{c} - 2af + g^2\right)$$

$$- \frac{fa + g^2}{a^2}(2\beta\gamma\omega_2\omega_3 + 2\gamma\alpha\omega_3\omega_1)$$

$$+ \frac{2\alpha\beta\omega_1\omega_2}{a^2}\left(2a\,\frac{af - g^2}{c} + g^2 - 2af\right).$$

Since, however, $c = 0$, every material particle of the body lies in the corresponding principal axis: it follows therefore that $\alpha = 0$, $\beta = 0$: hence, $\frac{af - g^2}{c}$ being finite,

$$\frac{R^2}{M^2} = \gamma^2\left(\frac{f}{a}\right)^2.$$

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But, putting $af = g^2$ in the expression for $\omega_1^2 + \omega_2^2$, we see that $\omega_1^2 + \omega_2^2 = \frac{f}{a}$: hence

$$\frac{R}{M} = \gamma (\omega_1^2 + \omega_2^2).$$

The hypothesis implied by the relation $c = 0$, is equivalent to limiting the general problem to the following case:

"A rigid body, every portion of which is immaterial, except a thin linear filament of which the mass is M, revolves about one of its points regarded as fixed: to find the pressure on the fixed point."

The result $R = M\gamma (\omega_1^2 + \omega_2^2)$

shews that the pressure is, as it plainly ought to be, independent of ω_3^2. In fact, the rigid body is substantially a straight rigid rod attached to a fixed point, and our general expression for R is a generalization of this simple case of centrifugal pressure.

The general expression for $\frac{R^2}{M^2}$ is of the form

$$\frac{R^2}{M^2} = A\omega_1^2 + B\omega_2^2 + C\omega_3^2 + 2A'\omega_2\omega_3 + 2B'\omega_3\omega_1 + 2C'\omega_1\omega_2 + \frac{u}{M^2},$$

whence, putting $\omega_1 = \omega \cos\lambda$, $\omega_2 = \omega \cos\mu$, $\omega_3 = \omega \cos\nu$, we have

$$\frac{R^2 - u}{M^2\omega^2} = A \cos^2\lambda + B \cos^2\mu + C \cos^2\nu + 2A' \cos\mu \cos\nu + 2B' \cos\nu \cos\lambda + 2C' \cos\lambda \cos\mu.$$

Thus $\frac{1}{\omega^2}(R^2 - u)$ varies as the square of the reciprocal of the radius vector ρ of the quadric surface

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy = D,$$

referred to the principal axes as axes of coordinates; and therefore

$$R^2 = u + v\left(\frac{\omega}{\rho}\right)^2,$$

where u and v are constants throughout the motion.

January 17, 1870.

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NOTE ON EUCLID, BOOK VI., PROP. 7.

By J. F. Wolff, Jun.

Mr. Walton, in No. 35 (March, 1868) of the *Quarterly Journal*, gives a different demonstration of the 7th Proposition of the 6th Book of Euclid.

There is still a more direct demonstration, which shews the whole bearing of the proposition and reduces it to a case of ambiguous equality.

Let the triangles ABC, DEF (fig. 9) have one angle of the one equal to one angle of the other; namely, $\angle BAC = \angle DEF$, and the sides about two other angles ABC, EDF proportionals, so that $AB : BC = ED : DF$.

In the triangle ABC, produce BC to any point L, and from the vertex A make AG equal to AC, and the angle GAL equal to the angle ABC.

From the point A make AM equal to DE, and from M draw MN parallel to BL.

Then $\angle BCA = \angle AGL$, and $\angle ABC = \angle GAL$.

Therefore the triangle AGL is similar to ABC.

But the triangle AGL is similar to the triangle AMO, because MN is parallel to BL.

Therefore the triangle ABC is similar to the triangle AMO, and $AB : BC = AM : AO$, but we have also

$$AB : BC = ED : DF.$$

Therefore $AM : AO = ED : DF$.

But $AM = DE$; therefore $AO = DF$.

AO is also equal to AN by construction, and the angle AMO being equal to BAC must also be equal to the angle DEF.

Therefore in the triangle AMO, as well as in the triangle AMN, we have two sides (AM and AO in the one, and AM and AN in the other) equal to the two sides DE and DF in the triangle DEF and adjacent angle AMN in each of the triangles equal to the adjacent angle DEF of the given triangle.

The question now arises whether the triangle DEF is equal to the triangle AMO, or to the triangle AMN. (We know already that AMO is similar to BAC).

The second condition given in Euclid discards all ambiguity in this case, and it is easily proved that the triangle AMO and not AMN must be equal to the triangle DEF.

If the remaining angle is a right angle, AN and AO, or AC and AG coincide; and the equality is obvious.

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THEOREM ON THE ACTION OF AN ELECTRIC CURRENT IN A HELIX WOUND ON A CYLINDER OF ANY FORM UPON THE POLE OF A MAGNET.

By Percival Frost, M.A.

Let N (fig. 10) be the north pole of a magnet acted on by a helix wrapped round a cylinder of any form, PQ an element of the helix, h the distance of the threads, NO perpendicular to the plane of the turn of which PQ is an element, the action of PQ takes place perpendicular to the plane QPN and is measured by $\frac{i\mu . PQ \sin QPN}{NP^2}$, where i is the intensity of the current and μ the magnetic fluid in N, and the component parallel to the axis of the cylinder by $\frac{i\mu . PQ \sin QPN}{NP^2} \cdot \frac{UO}{UN}$ if OU be perpendicular to PQ.

Consider the action of the portions of the wires lying in the rectangle $PP'Q'Q$, whose sides are parallel and perpendicular to the axis, the number of elements is $\frac{PP'}{h}$; therefore, the whole action of these portions is

$$\frac{i\mu . PQ . PP' \sin QPN . UO}{h . NP^2 . UN} .$$

With N as centre describe a sphere of radius a and project $PP'Q'Q$ upon it into $pp'q'q$. The inclination of the areas is that of OU and NP, which are normals to their planes; therefore

$$\text{area } PP'Q'Q . \frac{OU}{NP} : \text{area } pp'q'q :: NP^2 : a^2 ;$$

therefore the action parallel to the axis is

$$\frac{i\mu . NP \sin QPN}{ha^2 . UN} \times \text{area } pp'q'q = \frac{i\mu}{ha^2} \times \text{area } pp'q'q.$$

Hence the following curious theorem is true:

The whole action of a cylindrical helix on the pole of a magnet in the direction of the axis of the cylinder varies as the projection upon a sphere of all the parts of the cylinder on which the helix lies, doubled if they overlap.

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As an application of the theorem take the cyclinder circular and the pole in the axis.

The projection is the area generated by bc (fig. 11) round the axis

$$= 2\pi a^2 (\cos cNa - \cos bNa),$$

if r be the radius of the cylinder, $2l$ its length, the action on N

$$= \frac{2\pi i\mu}{h}\left[\frac{z+l}{\sqrt{\{r^2+(z+l)^2\}}} - \frac{z-l}{\sqrt{\{r^2+(z-l)^2\}}}\right],$$

$$= \frac{\pi i\mu n}{l}\left[\left\{1+\frac{r^2}{(z+l)^2}\right\}^{-\frac{1}{2}} + \left\{1+\frac{r^2}{(l-z)^2}\right\}^{-\frac{1}{2}}\right],$$

if n be the number of turns, or $nh = 2l$, and N be supposed to be within the cylinder; which is the formula used by Haedenkamp, *Pogg. Ann. Bd.* 78, 1849.

ON THE EVOLUTES OF CUBIC CURVES.

By Henry M. Jeffery, M.A.

1. The evolute of a curve of the second class ($u_2 = 0$) can be exprcssed in the form ($u_2 v_2 = 16\Delta^4$), if ($v_2 = 0$) denote the equation of the reciprocal curve (*Journal*, Vol. x., p. 332–336). It has been explained that this form exhibits the foci of the conic and its reciprocal as common to the evolute. Also, if $u_2 = 0 \equiv \mu_1\mu_2 + \kappa(4\Delta^2)$, $v_2 = 0 \equiv \mu_3\mu_4 + \lambda(4\Delta^2)$, so that $\mu_1\mu_2\mu_3\mu_4$ denote the foci, this conclusion readily appears from the equation to the evolute. In like manner, if ($u_3 = 0$) ($v_6 = 0$) denote the equations to a cubic curve of the third class, and its reciprocal curve (*continuous* for Spherical Geometry, but degenerating into points for Plane Geometry), the equation to its evolute may be written $u_3 v_6 + 16\Delta^4 . w_5 = 0$, if w_5 denote an auxiliary curve of the fifth class. It will also be shewn that a simple adaptation of this form will exhibit the evolute of a cubic of the third order.

2. To investigate the equation to the evolute of a non-singular cubic of the third class.

Let the equation be written in the canonical form

$$af^3 + bg^3 + ch^3 + 6dfgh = 0 \quad \ldots\ldots\ldots\ldots \quad (1).$$

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The equation to the evolute is the eliminant of this and two others, which denote the point of contact and the perpendicularity of the normal:

$$(af^2 + 2dgh)\,p + (bg^2 + 2dfh)\,q + (ch^2 + 2dfg)\,r = 0 \ldots (2),$$

$$fP + gQ + hR = 0 \ldots\ldots\ldots\ldots (3),$$

if P denote $a(ap - bq\cos C - cr\cos B)$, and Q, R are similarly interpreted, so that $pP + qQ + rR = 4\Delta^2$.

For the convenience of the analysis, write (1) in the form

$$AP^2f^2 + BQ^2g^2 + CR^2h^2 + 6DPQRfgh = 0 \ldots\ldots (4).$$

Also use the subsidiary quantities L, M, N, H thus defined:

$$L = APp + D(Pp - Qq - Rr)$$
$$M = BQq + D(Qq - Rr - Pp)$$
$$N = CRr + D(Rr - Pp - Qq)$$
$$-H = MN + NL + LM.$$

The following ratios of fP, gQ, hR may be obtained from (2) and (3),

$$fP(L - \sqrt{(H)} = gQ\{M + \sqrt{(H)}\},$$
$$gQ(M - \sqrt{(H)} = hR\{N + \sqrt{(H)}\},$$
$$hR(N - \sqrt{(H)} = fP\{L + \sqrt{(H)}\}.$$

Hence $$\frac{fP}{M + \sqrt{(H)}} = \frac{gQ}{L - \sqrt{(H)}} = -\frac{hR}{L + M}.$$

By substituting these ratios in (4)

$$-A\{(M + \sqrt{(H)}\}^3 - B\{L - \sqrt{(H)}\}^3 + C(L + M)^3$$
$$+ 6D\{M + \sqrt{(H)}\}\{L - \sqrt{(H)}\}(L + M) = 0.$$

By symmetry,

$$-B\{N + \sqrt{(H)}\}^3 - C\{M - \sqrt{(H)}\}^3 + A(M + N)^3$$
$$+ 6D\{N + \sqrt{(H)}\}\{M - \sqrt{(H)}\}(M + N) = 0$$
$$-C\{L + \sqrt{(H)}\}^3 - A\{N - \sqrt{(H)}\}^3 + B(N + L)^3$$
$$+ 6D\{L + \sqrt{(H)}\}\{N - \sqrt{(H)}\}(N + L) = 0.$$

By adding the three last equations

$$\sqrt{(H)}\{A(N^2 - M^2) + B(L^2 - N^2) + C(M^2 - L^2)\}$$
$$= A(M+N)(H - MN) + B(N+L)(H - NL) + C(L+M)(H - LM)$$
$$- 6D(M + N)(N + L)(L + M).$$

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to the

$Y^2+3H)$

$I-YZ$

$H-LM)\}$

which pre-

$-24.1BCD$

$-48D^2\}$

$-3(pq-CRr)$

Qq

their values $\frac{a}{P^2}, \frac{b}{Q^2}, \frac{c}{R^2}$,

$-24acd^2PQ^2R-24abd^2PQR^2$

$-16bd^2)-2P^2Q^2(abc^2+16cd^2)$

$acP^2R^2+acP^2Q^2)$

$2abd^2-2bcP^2-acQ^2)Qq$

$2bcP^2+2acQ^2-abR^2)Rr\}=0,$

$r\cos B)$ and (in Spherics)

tangential coordinate of
referred to the polar triangle

factor $(8d^2+abc)$ is the
the cubic. If the factor be
into the three given coincident

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3. To determine the equation to a non-singular cubic of the third order.

By interchanging the variables p, q, r, with P, Q, R in the resulting equation of § 2, we may obtain the required equation to the evolute of the cubic

$$a\alpha^3 + b\beta^3 + c\gamma^3 + 2da\beta\gamma = 0.$$

The principle of the process is best seen in Spherics, where both the primitive and reciprocal curves are finite.

4. To investigate the Boothian equation to the evolute of a non-singular cubic of the third class.

The equations to the curve and a point of contact, and the condition of perpendicularity of the normal, are

$$\phi(f, g) = 0, \quad (\xi - f)\frac{d\phi}{df} + (\eta - g)\frac{d\phi}{dg} = 0, \quad f\xi + g\eta = 0.$$

Their eliminant determines the evolute.

A non-singular cubic of the third class is thus denoted:

$$af^3 + bg^3 + c + 6dfg = 0.$$

The equation to its evolute is

$$\begin{aligned} &4d^3(\xi^3 - \eta^3)^3 - 3cd\xi\eta(\xi^3 - \eta^3)(b\xi + a\eta) + ac^3\eta^3 - bc^3\xi^3 \\ &= \{4d^3(\xi^3 - \eta^3)^3 - 4cd\xi\eta - c\xi\eta(a\eta + b\xi)\} \\ &\times \sqrt{\{d^3(\xi^3 - \eta^3)^3 + 2cd\xi\eta - c\xi\eta(a\eta + b\xi)\}}. \end{aligned}$$

This may be otherwise expressed:

$$\begin{aligned} &(a\xi^3 + b\eta^3 + c + 6d\xi\eta)\{b^2c^2\xi^6 - 2(abc + 16d^3)\xi^3\eta^3 + a^2c\eta^6\} \\ &= (\xi^3 + \eta^3)^2(8d^3 + abc)\{-6d\xi^3\eta^3 + a\eta^3(\eta^3 - 2\xi^3) + b\xi^3(\xi^3 - 2\eta^3)\}. \end{aligned}$$

Cheltenham, *June* 1, 1870.

(To be continued).

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NOTE ON A RELATION BETWEEN TWO CIRCLES.

By Professor CAYLEY.

CONSIDER any two circles O, Q; and let AC, BD, $A'D'$, $B'C'$ (fig. 12) be the common tangents touching the circles in the points A, A', B, B', C, C', D, D': the locus of a point P such that the pairs of tangents from it to the two circles respectively form a harmonic pencil, is a conic through the 8 points A, A', B, B', C, C', D, D'; but this conic may break up into two lines, viz., if (as in the figure) the points A, B', D', D are in a line, then the points C, C', A', B, will be in a symmetrically situated line, and the conic breaks up into this pair of lines, meeting, suppose in K. The condition for this, if a, a' are the distances of the centres from a fixed point in the line of centres, and if the radii are c, c' is readily found to be

$$(a - a')^2 = 2(c^2 + c'^2).$$

Suppose in general, that (given any two conics) the point P' is the intersection of the polars of P in regard to the two given conics respectively; then if P describes a line, the locus of P' is a conic passing through the three conjugate points of the given conics; if, however, the line which is the locus of P pass through one of the conjugate points, then the conic the locus of P' breaks up into a pair of lines, one of them a fixed line through the other two conjugate points, the other of them a line through the first mentioned conjugate point. That is, if the locus of P be a line through a conjugate point, the locus of P' is a line through the same conjugate point; but in every other case the locus of P' is a conic.

Reverting to the figure of the two circles, in order that it may be possible that the two lines AD and BC may be loci of points P, P', related as above, it is necessary that K shall be a conjugate point of the two circles; that is, if the two circles intersect in points Λ, Λ' lying symmetrically in the radical axis, which meets, suppose, the line of centres in M, then it is necessary that K shall be one of the anti-points of Λ, Λ'; or what is the same thing, the

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distance KM must be $= i$ into $M\Lambda$ or $M\Lambda'$; this condition, if as above $(a-a')^2 = 2(c^2+c'^2)$, implies $c^2 = c'^2$, and we have then $(a-a')^2 = 4c^2$, that is, the circles must be equal, and the distance of the centres must be twice the radius, or what is the same thing, the circles must be equal circles touching each other; when this is so, the two lines AD, BC being then lines at right angles to each other intersecting in the point of contact, have, in fact, the above mentioned relation. And it thus appears that given two circles, the necessary and sufficient conditions for the coexistence of the properties mentioned in the theorem are that they shall be equal circles touching each other.

ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED POLYGON, AND THE (2, 2) CORRESPONDENCE OF POINTS ON A CONIC.

By Prof. CAYLEY.

THE present paper includes, as will at once be seen, much that is perfectly well known; but the separate theories required, it seemed to me, to be put together; and there are, particularly as regards the unsymmetrical case afterwards referred to, some results which I believe to be new.

The porism of the in-and-circumscribed polygon has its foundation in the theory of the symmetrical (2, 2) correspondence of points on a conic; viz. a (2, 2) correspondence is such that to any given position of either point there correspond two positions of the other point; and in a symmetrical (2, 2) correspondence either point indifferently may be considered as the first point and the other of them will then be the second point of the correspondence. Or, what is the same thing, if x, y are the parameters which serve to determine the two points, then x, y are connected by an equation of the form $(*)(x, 1)^2(y, 1)^2 = 0$, which is symmetrical in regard to the two parameters (x, y). In the case of such symmetrical relation it is easy to show that the line joining the two points envelopes a conic. For the relation may be expressed in the form $(*\chi 1, x+y, xy)^2 = 0$;

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we may imagine the coordinates (P, Q, R) fixed in such manner that for the point (x) on the first conic we have $P:Q:R=1:x:x^2$, and for the point (y), $P:Q:R=1:y:y^2$; the equation of the line joining the two points is, then,

$$\begin{vmatrix} P, & Q, & R \\ 1, & x, & x^2 \\ 1, & y, & y^2 \end{vmatrix} = 0;$$

that is

$$Pxy - Q(x+y) + R = 0,$$

or representing this by

$$P\xi + Q\eta + R\zeta = 0,$$

we have $\xi:\eta:\zeta = xy:-x-y:1$; and consequently (ξ, η, ζ) are connected by a quadric equation; that is, the envelope is a conic.

The relation $(*\mathfrak{X}x, 1)^2(y, 1)^2 = 0$, whether symmetrical or not, leads as will be presently shown to a differential equation of the form

$$\frac{dx}{\sqrt{(X)}} \pm \frac{dy}{\sqrt{(Y)}} = 0,$$

where X, Y are quartic functions of x, y respectively; viz., these are unlike or like functions of the two variables according as the integral equation is not or is symmetrical in regard to the two variables. In the former case, however, the functions X, Y are so related to each other, that the two can be by a linear transformation converted into like functions of the variables: for instance, if y be changed into $ay_1 + b \div cy_1 + d$, then the constants may be determined in suchwise that Y is the same function of y_1, that X is of x; the original integral equation being hereby converted into a symmetrical equation $(*\mathfrak{X}x, 1)^2(y_1, 1)^2 = 0$ between x and y_1, so that in one point of view the unsymmetrical case is not really more general than the symmetrical one. It is to be added that the integral equation contains really one more constant than the differential equation (this is most readily seen in the symmetrical case, the differential equation depends only on the ratio of five constants a, b, c, d, e, whereas the integral equation depends on the ratio of six constants) so that the integral equation is really the complete integral of the differential equation.

Attending now to the symmetrical case; if A and B are corresponding points, then the corresponding points of B are A and a new point C; those of C are B and a new point

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D, and so on; so that the points form a series $A, B, C, D\ldots$; and the porismatic property is that, if for a given position of A this series closes at a certain term, for instance, if $D = A$; then it will always thus close, whatever be the position of A. And this follows at once from the consideration of the differential equation $\frac{dx}{\sqrt{(X)}} = \frac{dy}{\sqrt{(Y)}}$; viz., as this is at once integrable *per se* in the form

$$\Pi(y) - \Pi(x) = \Pi(k),$$

this equation must be a transformation of the original equation $(*\mathfrak{X}x, 1)^2(y, 1)^2 = 0$, and equally with it represent the relation betwen the parameters x, y of the two points A, B; the constant of integration k is of course completely determined in terms of the coefficients of the last mentioned equation, assumed to be given.

Hence forming the equations for the correspondences, B, C; $C, D\ldots$ and assuming that the series closes F, A; we have

$$\Pi(z) - \Pi(y) = \Pi(k),$$
$$\vdots$$
$$\Pi(x) - \Pi(u) = \Pi(k);$$

where, however, the $\Pi(x)$ of the last equation must be regarded as differing from that of the first equation by a period, say Ω, of the integral; hence adding, we have

$$\Omega = n\Pi(k),$$

or

$$\Pi(k) = \frac{1}{n}\Omega,$$

which gives between the constants of the integral equation $(*\mathfrak{X}x, 1)^2(y, 1)^2 = 0$, a relation which must be satisfied when the series, closes at the n^{th} term (viz. when the term after this coincides with the first term); and this relation is independent of x, that is, of the position of the point A.

The analysis in regard to the differential equation is as follows:

Consider the equation

$$\begin{aligned} U = {} & y^2(ax^2 + 2bx + c) \\ & + 2y(a'x^2 + 2b'x + c') \\ & + (a''x^2 + 2b''x + c'') = 0, \end{aligned}$$

say $U = (P, Q, R\mathfrak{X}y, 1)^2 = (L, M, N\mathfrak{X}x, 1)^2 = 0,$

we have $dU = 0 = (Py + Q)\,dy + (Lx + M)\,dx.$

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But the equation $U=0$ gives

$$(Py+Q)^2 = Q^2 - PR,$$
$$(Lx+M)^2 = M^2 - NL,$$

and the differential equation therefore becomes

$$dy\,\sqrt{(Q^2-PR)} \pm dx\,\sqrt{(M^2-NL)} = 0,$$

viz. it is

$$\frac{dy}{\sqrt{\{(ay^2+2a'y+a'')\,(cy^2+2c'y+c'')-(bx^2+2b'y+b'')^2\}}}$$
$$\pm \frac{dx}{\sqrt{\{(ax^2+2bx+c)\,(a''x^2+2b''x+c'')-(a'x^2+2b'x+c')^2\}}} = 0.$$

Suppose the equation is

$$\begin{aligned} & y^2\,(ax^2+2hx+g) \\ & +2y\,(hx^2+2bx+f) \\ & +\quad (gx^2+2fx+c) = 0, \end{aligned}$$

then the differential equation is

$$\frac{dy}{\sqrt{\{(ay^2+2hy+g)\,gy^2+2fy+c)^2-(hy^2+2by+f)^2\}}}$$
$$\pm \frac{dx}{\sqrt{\{(ax^2+2hx+g)\,(gx^2+2fx+c)-(hx^2+2bx+f)^2\}}} = 0,$$

say

$$\frac{dy}{\sqrt{(Y)}} = \frac{\pm dx}{\sqrt{(X)}}.$$

Now starting from the differential equation

$$\frac{dx}{\sqrt{\{(a,\,b,\,c,\,d,\,e\mathcal{X}(x,\,1)^4\}}} = \pm \frac{dy}{\sqrt{\{(a,\,b,\,c,\,d,\,e\mathcal{X}(y,\,1)^4\}}},$$

the integral equation is known to be

$$\left[\frac{\sqrt{\{(a,\,b,\,c,\,d,\,e\mathcal{X}(x,\,1)^4\}}-\sqrt{\{(a,\,b,\,c,\,d,\,e\mathcal{X}(y,\,1)^4\}}}{x-y}\right]$$
$$= a\,(x+y)^2+4b\,(x+y)+6\theta,$$

where θ is the constant of integration. Writing, for shortness, $X=(a,\,b,\,c,\,d,\,e\mathcal{X}(x,\,1)^4$, $Y=(a,\,b,\,c,\,d,\,e\mathcal{X}(y,\,1)^4$, this is

$$X+Y-2\,\sqrt{(XY)} = a\,(x^2-y^2)^2+4b\,(x-y)\,(x^3-y^3)+6\theta(x-y)^2;$$

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or, what is the same thing,

$$a(x^4+y^4)-2\sqrt{(XY)}=a(x^2-y^2)^2+4b(x-y)(x^2-y^2)+6\theta(x-y)^2,$$

$$+4b\,(x^3+y^3)$$
$$+6c\,(x^2+y^2)$$
$$+4d(x+y)$$
$$+\quad 2e$$

viz. this gives

$$\begin{aligned}\sqrt{(XY)} = \quad & ax^2y^2 \\ & +2b\,(x^2y+xy^2) \\ & +3c\,(x^2+y^2) \\ & +3\theta\,(x-y)^2 \\ & +2d\,(x+y) \\ & +e,\end{aligned}$$

and rationalising, the integral equation becomes

$$\begin{aligned} & -6a\theta x^2y^2 \\ & -4adxy\,(x+y) \\ & -ae\,(x+y)^2 \\ & +4b^2x^2y^2 \\ & +12bcxy\,(x+y)-12b\theta xy\,(x+y) \\ & -8bdxy \\ & -4be\,(x+y) \\ & +9c^2\,(x+y)^2-18c\theta\,(x^2+y^2) \\ & -12cd\,(x+y) \\ & +9\theta^2\,(x-y)^2-12d\theta\,(x+y)-6e\theta+4d^2=0; \end{aligned}$$

or, as it may be written,

$$\begin{aligned} & x^2y^2\,(4b^2-6a\theta) \\ & +(x^2y+xy^2)\,(-4ad+12bc-12b\theta) \\ & +\quad (x^2+y^2)\quad (-ae+9c^2-18c\theta+9\theta^2) \\ & +\qquad xy\qquad (-2ae-8bd+18c^2-18\theta^2) \\ & +\quad (x+y)\quad (-4be+12cd-12d\theta) \\ & +\qquad\qquad 4d^2-6e\theta=0. \end{aligned}$$

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Comparing this with the original integral equation $V=0$, and the form of differential equation deduced therefrom, we ought to have identically

$$[(4b^2-6a\theta)x^2+(-2ad+6bc-6b\theta)x+(-ae+9c^2-18c\theta+9\theta^2)]$$
$$\times[(-ae+9c^2-18c\theta+9\theta^2)x^2+(-2be+6cd-6d\theta)x+(4d^2-6e\theta)]$$
$$-[(-2ad+6bc-6b\theta)x^2+(-ae-4bd+9c^2-9\theta^2)x$$
$$+(-2be+6cd-6d\theta)]^2$$
$$=\text{multiple of } X$$
$$=\{(-4ad^2-4b^2e+24bcd)+(6ae-24bd-54c^2)\theta+108c\theta^2-54\theta^3\}$$
$$\times(a, b, c, d, e\mathcal{X}x, 1)^4,$$

by comparing the coefficients of x^4.

I obtain this otherwise.

Write $$V=\alpha U+6\beta H,$$

then, forming the Hessian of V, we have

$$\tilde{H}V=(\alpha^2-3I\beta^2)H+(I\alpha\beta+9J\beta^2)U$$
$$=\frac{(\alpha^2-3I\beta^2)}{6\beta}(V-\alpha U)+(I\alpha\beta+9J\beta^2)U$$
$$=\frac{\alpha^2-3I\beta^2}{6\beta}V+\frac{1}{6\beta}(-\alpha^3+9I\alpha\beta^2+54J\beta^3)U,$$

that is

$$d_x^2Vd_y^2V-(d_xd_yV)^2-\frac{2(\alpha^2-3I\beta^2)}{\beta}(x^2d_x^2V+2xyd_xd_yV+y^2d_y^2V)$$
$$=\frac{24}{\beta}(-\alpha^3+9I\alpha\beta^2+54J\beta^3)U,$$

or writing $$K=-\frac{2(\alpha^2-3I\beta^2)}{\beta},$$

this is $$(d_x^2V+Ky^2)(d_y^2V+Kx^2)-(d_xd_yV-Kxy)^2$$
$$=\frac{24}{\beta}(-\alpha^3+9I\alpha\beta^2+54J\beta^3)U,$$

so that the components are

$$d_x^2V+Ky^2,\ d_xd_yV-Kxy,\ d_y^2V+Kx^2,$$

$$V=\alpha U+6\beta H=\alpha(a, b, c, d, e\mathcal{X}x, 1)^4$$
$$+6\beta(ac-b^2, 2ad-2bc, ae+2bd-3c^2, 2be-2cd, ce-d^2)\dagger(x, 1)^4,$$

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viz. components are

$$(\alpha\alpha+6\beta(ac-b^2), \alpha b+3\beta(ad-bc), ac+\beta(ae+2bd-3c^2)+\tfrac{1}{12}K\,(x,1)^2,$$

$$(\alpha b+3\beta(ad-bc), ac+\beta(ae+2bd-3c^2)+\tfrac{1}{24}K, \alpha d+3\beta(be-cd)\,(x,1)^2,$$

$$(ac+\beta(ae+2bd-3c^2)+\tfrac{1}{12}K, \alpha d+3\beta(be-cd), \alpha e+6\beta(ce-d^2)\,(x,1)^2,$$

$$K=-\frac{2(\alpha^2-3I\beta^2)}{\beta}.$$

I assume $\beta=-\frac{2}{3}$, $\alpha=4c-6\theta$, $K=3\{(4c-6\theta)^2-\frac{4}{3}I\}$.

$$\alpha\alpha+6\beta(ac-b^2)=a(4c-6\theta)-4(ac-b^2)=4b^2-6a\theta,$$

$$\alpha b+3\beta(ad-bc)=b(4c-6\theta)-2(ad-bc)=-2ad+6bc-6b\theta,$$

$$ad+3\beta(be-cd)=d(4c-6\theta)-2(be-cd)=-2be+6cd-6d\theta,$$

$$ae+6\beta(ce-d^2)=e(4c-6\theta)-4(ce-d^2)=4d^2-6e\theta,$$

$$ac+\beta(ae+2bd-3c^2)-\tfrac{1}{24}K$$
$$=c(4c-6\theta)-\tfrac{2}{3}(ae+2bd-3c^2)-\tfrac{1}{8}\{(4c-6\theta)^2-\tfrac{4}{3}I\}$$
$$=-\tfrac{1}{2}ae-2bd+\tfrac{8}{3}c^2-\tfrac{8}{3}\theta^2,$$

$$ac+\beta(ae+2bd-3c^2)+\tfrac{1}{12}K$$
$$=c(4c-6\theta)-\tfrac{2}{3}(ae+2bd-3c^2)+\tfrac{1}{4}\{(4c-6\theta)^2-\tfrac{4}{3}I\}$$
$$=-ae+9c^2-18c\theta+9\theta^2,$$

agreeing with the former result.

I return to the general form

$$y^2(a,\ b,\ c\,(x,1)^2$$
$$+2y(a',\ b',\ c'(x,1)^2$$
$$+\quad(a'',\ b'',\ c''(x,1)^2=0,$$

giving

$$\frac{dx}{\sqrt{[(a,\ b,\ c(x,1)^2\,(a'',\ b'',\ c''(x,1)^2-\{(a',\ b',\ c'(x,1)^2\}^2]}}$$
$$=\frac{dy}{\sqrt{[(a,\ a',\ a''(y,1)^2\,(c,\ c',\ c''(y,1)^2-\{(b,\ b',\ b''(y,1)^2\}^2]}}.$$

Operate a linear transformation on the x, say

$$x=\frac{\lambda x'+\mu}{\nu x'+\rho};$$

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the new coefficients are

$$(a,\ b,\ c\ \mathfrak{x}\lambda,\nu)^2,\ (a,\ b,\ c\ \mathfrak{x}\lambda,\nu\mathfrak{x}\mu,\rho),\ (a,\ b,\ c\ \mathfrak{x}\mu,\rho)^2,$$
$$(a',\ b',\ c'\mathfrak{x}\lambda,\nu)^2,\ (a',\ b',\ c'\mathfrak{x}\lambda,\nu\mathfrak{x}\mu,\rho),\ (a',\ b',\ c'\mathfrak{x}\mu,\rho)^2,$$
$$(a'',\ b'',\ c''\mathfrak{x}\lambda,\nu)^2,\ (a'',\ b'',\ c''\mathfrak{x}\lambda,\nu\mathfrak{x}\mu,\rho),\ (a'',\ b'',\ c''\mathfrak{x}\mu,\rho)^2:$$

assume now

$$\begin{aligned} a'\lambda &+(b'-\theta)\nu- & a\mu &- & b\rho &=0,\\ (b'+\theta)\lambda+ & & c'\nu &- & b\mu &- & c\rho &=0,\\ a''\lambda &+ & b''\nu &- & a'\mu &-(b'+\theta)\rho=0,\\ b''\lambda &+ & c''\nu &-(b'-\theta)\mu- & c'\rho &=0, \end{aligned}$$

then it is easy to show that

$$\begin{aligned} (a,\ b,\ c\ \mathfrak{x}\lambda,\nu\mathfrak{x}\mu,\rho) &= (a',\ b',\ c'\mathfrak{x}\lambda,\nu)^2,\\ (a',\ b',\ c'\mathfrak{x}\mu,\rho)^2 &= (a'',\ b'',\ c''\mathfrak{x}\lambda,\nu\mathfrak{x}\mu,\rho),\\ (a,\ b,\ c\ \mathfrak{x}\mu,\rho)^2 &= (a'',\ b'',\ c''\mathfrak{x}\lambda,\nu)^2\\ &[=(a',\ b',\ c'\mathfrak{x}\lambda,\nu\mathfrak{x}\mu,\rho)+\theta(\lambda\rho-\mu\nu)], \end{aligned}$$

and the equations give

$$\begin{vmatrix} a' & , & b'-\theta, & a & , & b\\ b'+\theta, & & c' & , & b & , & c\\ a'' & , & b'' & , & a' & , & b'+\theta\\ b'' & , & c'' & , & b'-\theta, & & c' \end{vmatrix} =0,$$

that is
$$\begin{aligned} &(a'c'-b'^2+\theta^2)^2+(a''c''-b''^2)(ac-b^2),\\ &-(a'b''-a''b'+a''\theta)(bc'-b'c+c\theta)\\ &+(a'c''-b'b''+b''\theta)(bb'-a'c+b\theta)\\ &+(b'b''-a''c'+b''\theta)(ac'-bb'+b\theta)\\ &-(b'c''-b''c'+c''\theta)(ab'-a'b+a\theta)=0, \end{aligned}$$

which is

$$\begin{aligned} &(a'c'-b'^2)^2+(a''c''-b''^2)(ac-b^2) && -2\theta \begin{vmatrix} a, & b, & c\\ a', & b', & c'\\ a'', & b'', & c'' \end{vmatrix}\\ &+a'^2(-cc'')\\ &+b'^2(-ac''-2bb''-a''c)\\ &+c'^2(-aa'') && +\theta^2(2(a'c'-b'^2)-ac''+2bb''-a''c)\\ &+2b'c'(ab''+a''b)\\ &+2c'a'(-bb'')\\ &+2a'b'(bc''+b''c) && +\theta^4=0. \end{aligned}$$

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If the original matrix be symmetrical $= \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$, this is

$$\begin{aligned} &(fh-b^2)^2+(ag-h^2)(cg-f^2) && -2\theta \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \\ &+h^2(-cg) \\ &+b^2(-ac-g^2-2fh) \\ &+f^2(-ag) \\ &+2bf(af+gh) && +\theta^2[2(fh-b^2)-ac-g^2+2hf) \\ &+2fh(-fh) \\ &+2bh(fg+ch) && +\theta^4=0, \end{aligned}$$

that is

$$(b-g)\{(b^2-ac)(b+g)+2(af^2+ch^2-2bfh)\}$$
$$-2\theta(abc-af^2-bg^2-ch^2+2fgh)+\theta^2(4fh-2b^2-ac-g^2)+\theta^4=0,$$

satisfied by

$$\theta+b-g=0,$$

viz. the equation in θ is

$$(\theta+b-g)\{\theta^2-(b-g)\theta^2+(4fh-ac-2bg-b^2)\theta$$
$$+(b^2-ac)(b+g)+2(af^2+ch^2-2bfh)\}=0.$$

NOTE ON RHIZIC CURVES.

By WILLIAM WALTON, M.A., Fellow of Trinity Hall.

LET $f(x)$ represent the expression

$$x^m+p_1x^{m-1}+p_2x^{m-2}+p_3x^{m-3}+\ldots+p_m,$$

where p_1, p_2, p_3,$\ldots p_m$, are all possible constants. Write $u+v\sqrt{(-1)}$ for x, where u and v are possible quantities: then

$$f(x)=P+Q\sqrt{(-1)},$$

P and Q being possible functions of u and v. The two curves $P=0$, $Q=0$, referred to rectangular axes of coordinates u and v, I propose to call rhizic curves, in virtue of their relation to the roots of the equation $f(x)=0$, and I proceed to investigate some of the properties of these curves.

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Since the equation $f(x)=0$ has m roots, the rhizic curves intersect each other at m points. Let θ, ϕ, be the inclinations of the two curves, at a point of intersection, to the axis of u. Then

$$\frac{dP}{du}\cos\theta+\frac{dP}{dv}\sin\theta=0 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

and

$$\frac{dQ}{du}\cos\phi+\frac{dQ}{dv}\sin\phi=0 \quad \ldots\ldots\ldots\ldots\ldots (2).$$

But (I refer to my article on a Theorem in Maxima and Minima, *Quarterly Journal*, Vol. x., p. 253) we know that $\frac{dQ}{du}=-\frac{dP}{dv}$, and $\frac{dQ}{dv}=\frac{dP}{du}$; hence the equation (2) becomes

$$-\frac{dP}{dv}\cos\phi+\frac{dP}{du}\sin\phi=0 \quad \ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (3) we see that $\cos(\theta-\phi)=0$, and consequently that, when there is no multiplicity at an intersection,

I. "*The rhizic curves intersect each other always at right angles.*"

Imagine that (u, v) is a multiple point, of ν^{th} multiplicity, of the curve $P=0$; then

$$\left(\cos\theta\frac{d}{du}+\sin\theta\frac{d}{dv}\right)^{\nu}P=0,$$

will determine the directions of all its tangents at this point. Suppose the multiplicity to be even, and put $\nu=2r$. Then, as may be seen by reference to the article quoted above,

$$\frac{d^{2r}P}{du^{2r}}\cos 2r\theta-\frac{d^{2r}Q}{du^{2r}}\sin 2r\theta=0.$$

Let α be such an angle that

$$\tan\alpha=\frac{\dfrac{d^{2r}P}{du^{2r}}}{\dfrac{d^{2r}Q}{du^{2r}}}.$$

Then the directions of the tangents at the multiple point are determined by the $2r$ values of θ, viz.,

$$\frac{\alpha}{2r}, \frac{\pi+\alpha}{2r}, \frac{2\pi+\alpha}{2r}, \frac{3\pi+\alpha}{2r}, \ldots \frac{(2r-1)\pi+\alpha}{2r}.$$

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In like manner, if the multiplicity be even, the equation for multiple tangency for the curve $Q=0$ will be, by the said article,

$$\frac{d^{\nu}Q}{du^{\nu}}\cos 2r\phi+\frac{d^{\nu}P}{du^{\nu}}\sin 2r\phi=0,$$

and therefore

$$\tan 2r\phi=-\cot\alpha=\tan\left(\frac{\pi}{2}+\alpha\right),$$

the directions of the tangents being therefore determined by $2r$ values of ϕ, viz.,

$$\frac{\frac{\pi}{2}+\alpha}{2r},\ \frac{\pi+\frac{\pi}{2}+\alpha}{2r},\ \frac{2\pi+\frac{\pi}{2}+\alpha}{2r},\ \frac{3\pi+\frac{\pi}{2}+\alpha}{2r},\ \ldots\ \frac{(2r-1)\pi+\frac{\pi}{2}+\alpha}{2r}.$$

Consequently,

II. "*When a rhizic curve has even multiplicity at any point, the multiplicity is equiangular.*"

III. "*When the two curves intersect at a point of even multiplicity,* the rotation, in the plane of the curves, of the tangents to either of them at the point, through an angle* $\frac{\pi}{4r}$, *will bring these tangents into coincidence with the tangents to the other at the point.*"

Next, let us suppose that there is odd multiplicity at any point (u, v) of the curve $P=0$, and put accordingly $\nu=2r+1$. Then the tangency is given by the equation

$$\left(h\frac{d}{du}+k\frac{d}{du}\right)^{2r+1}P=0,$$

$u+h$, $v+k$, being the coordinates of a point of the curve indefinitely near to (u, v).

It is easily seen, by reference to the quoted article, that

$$\mu \text{ even}, \ \ldots\frac{d^{2r+1}P}{du^{2r-\mu+1}dv^{\mu}}=(-1)^{\frac{\mu}{2}}\frac{d^{2r+1}P}{du^{2r+1}},$$

and, $$\mu \text{ odd}, \ \ldots\frac{d^{2r+1}P}{du^{2r-\mu+1}dv^{\mu}}=(-1)^{\frac{\mu+1}{2}}\frac{d^{2r+1}Q}{du^{2r+1}},$$

* If the multiplicity of one curve at an intersection of the two be even, that of the other is also even and of the same degree.

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Hence the equation of tangency becomes

$$h^{2r+1}\frac{d^{2r+1}P}{du^{2r+1}} - \frac{2r+1}{1}h^{2r}k\frac{d^{2r+1}Q}{du^{2r+1}}$$
$$- \frac{(2r+1).2r}{1.2}h^{2r-1}k^2\frac{d^{2r+1}P}{du^{2r+1}}$$
$$+ \frac{(2r+1).2r.(2r-1)}{1.2.3}h^{2r-2}k^3\frac{d^{2r+1}Q}{du^{2r+1}}$$
$$+ \frac{(2r+1).2r.(2r-1)(2r-2)}{1.2.3.4}h^{2r-3}k^4\frac{d^{2r+1}P}{du^{2r+1}}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ (-)^r\frac{(2r+1).2r}{1.2}h^2k^{2r-1}\frac{d^{2r+1}Q}{du^{2r+1}}$$
$$+ (-)^r\frac{2r+1}{1}hk^{2r}\frac{d^{2r+1}P}{du^{2r+1}}$$
$$+ (-)^{r+1}k^{2r+1}\frac{d^{2r+1}Q}{du^{2r+1}} = 0,$$

which is equivalent to

$$\frac{d^{2r+1}P}{du^{2r+1}}\left\{h^{2r+1} - \frac{(2r+1).2r}{1.2}h^{2r-1}k^2\right.$$
$$\left. + \frac{(2r+1).2r.(2r-1).(2r-2)}{1.2.3.4}h^{2r-3}k^4\right.$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\left. + (-1)^r\frac{2r+1}{1}hk^{2r}\right\}$$

$$\frac{d^{2r+1}Q}{du^{2r+1}}\left\{\frac{2r+1}{1}h^{2r}k - \frac{(2r+1).2r.(2r-1)}{1.2.3}h^{2r-2}k^3 + \ldots\right.$$
$$\left. + (-1)^{r+1}\frac{(2r+1).2r}{1.2}h^2k^{2r-1}\right.$$
$$\left. + (-1)^{r+2}k^{2r+1}\right\} = 0,$$

$$\frac{d^{2r+1}P}{du^{2r+1}}[h^{2r+1} + \{h - k\sqrt{(-1)}\}^{2r+1}]$$
$$= \sqrt{(-1)}\frac{d^{2r+1}Q}{du^{2r+1}}[h^{2r+1} - \{h - k\sqrt{(-1)}\}^{2r+1}] = 0,$$

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which, if h, k, be replaced by $\rho\cos\theta$, $\rho\sin\theta$, respectively, becomes

$$\frac{d^{2r+1}P}{du^{2r+1}}\cos(2r+1)\theta-\frac{d^{2r+1}Q}{du^{2r+1}}\sin(2r+1)\theta=0.$$

Thus, if $$\frac{\dfrac{d^{2r+1}P}{du^{2r+1}}}{\dfrac{d^{2r+1}Q}{du^{2r+1}}}=\tan\alpha,$$

the attitudes of the tangents at the point are given by the following $2r+1$ values of θ, viz.,

$$\frac{\alpha}{2r+1},\ \frac{\pi+\alpha}{2r+1},\ \frac{2\pi+\alpha}{2r+1},\ \cdots\ \frac{2r\pi+\alpha}{2r+1}.$$

Thus we see that

IV. "*When a rhizic curve has a point of odd multiplicity, the multiplicity is equiangular.*"

Let us now consider odd multiplicity in the curve $Q=0$, the equation of tangency being

$$\left(h\frac{d}{du}+k\frac{d}{dv}\right)^{2r+1}Q=0.$$

Now, it is easily seen that

$$\mu \text{ even}, \ \cdots\ \frac{d^{2r+1}Q}{du^{2r-\mu+1}dv^{\mu}}=(-1)^{\frac{1}{2}\mu}\ \frac{d^{2r+1}Q}{du^{2r+1}},$$

and, $$\mu \text{ odd}, \ \cdots\ \frac{d^{2r+1}Q}{du^{2r-\mu+1}dv^{\mu}}=(-1)^{\frac{1}{2}(\mu+1)}\frac{d^{2r+1}P}{du^{2r+1}}.$$

Hence the equation of tangency becomes

$$\begin{aligned}
&h^{2r+1}\frac{d^{2r+1}Q}{du^{2r+1}}+\frac{2r+1}{1}h^{2r}k\frac{d^{2r+1}P}{du^{2r+1}}\\
&\quad-\frac{(2r+1).2r}{1.2}h^{2r-1}k^{2}\frac{d^{2r+1}Q}{du^{2r+1}}\\
&\quad-\frac{(2r+1).2r.(2r-1)}{1.2.3}h^{2r-2}k^{3}\frac{d^{2r+1}P}{du^{2r+1}}\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad+(-1)^{r-1}\frac{(2r+1).(2r)}{1.2}h^{2}k^{2r-1}\frac{d^{2r+1}P}{du^{2r+1}}\\
&\quad+(-1)^{r}\frac{2r+1}{1}hk^{2r}\frac{d^{2r+1}Q}{du^{2r+1}}\\
&\quad+(-1)^{r}k^{2r+1}\frac{d^{2r+1}P}{du^{2r+1}}=0,
\end{aligned}$$

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which may be written

$$\frac{d^{2r+1}Q}{du^{2r+1}}\left\{h^{2r+1}-\frac{(2r+1).2r}{1.2}h^{2r-1}k^2+\ldots\right.$$
$$\left.+(-1)^r\frac{2r+1}{1}hk^{2r}\right\}$$
$$+\frac{d^{2r+1}P}{du^{2r+1}}\left\{\frac{2r+1}{1}h^{2r}k-\frac{(2r+1).2r.(2r-1)}{1.2.3}h^{2r-2}k^3\right.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+(-1)^{r-1}\frac{(2r+1).2r}{1.2}h^2k^{2r-1}$$
$$\left.+(-1)^r k^{2r+1}\right\}=0,$$

and therefore, ϕ denoting the angle of tangency,

$$\frac{d^{2r+1}Q}{du^{2r+1}}\cos(2r+1)\phi+\frac{d^{2r+1}P}{du^{2r+1}}\sin(2r+1)\phi=0.$$

Hence, if
$$\frac{\dfrac{d^{2r+1}P}{du^{2r+1}}}{\dfrac{d^{2r+1}Q}{du^{2r+1}}}=\tan\alpha,$$

we see that

$$\tan(2r+1)\phi=-\cot\alpha=\tan\left(\frac{\pi}{2}+\alpha\right),$$

and consequently the positions of the tangents are given by the $2r+1$ values of ϕ, viz.

$$\frac{\frac{1}{2}\pi+\alpha}{2r+1},\ \frac{\pi+\frac{1}{2}\pi+\alpha}{2r+1},\ \frac{2\pi+\frac{1}{2}\pi+\alpha}{2r+1},\ \ldots\ \frac{2r\pi+\frac{1}{2}\pi+\alpha}{2r+1}.$$

Thus we see that

V. "*When two rhizic curves intersect at a point of odd multiplicity,* the proposition* III., *stated in relation to even multiplicity, still holds good,* $2r$ *being replaced by* $2r+1$.

Prof. Cayley's addition (*Quarterly Journal*, Vol. x., p. 262) to my article on a Theorem in Maxima and Minima, makes

* If the multiplicity of one curve, at an intersection of the two, be odd, that of the other is also odd and of the same degree.

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it evident that the question was not left in its complete state by me. What he has shewn in relation to the particular case of double points may be generalized for all even multiplicities.

Take my formula, p. 257, viz.

$$\lfloor 2r\,(P'-P) = \rho^{2r}\left\{\frac{d^{2r}P}{du^{2r}}\cos 2r\theta - \frac{d^{2r}Q}{du^{2r}}\sin 2r\theta\right\} + \&c. \ldots\ldots (A).$$

Take also, (from p. 256),

$$\frac{d^{2r}P}{du^{2r}}\sin 2r\theta + \frac{d^{2r}Q}{du^{2r}}\cos 2r\theta = 0 \ldots\ldots\ldots\ldots (B).$$

1st. Let $\frac{d^{2r}P}{du^{2r}} = 0$. Then

$$\lfloor 2r\,(P'-P) = -\rho^{2r}\frac{d^{2r}Q}{du^{2r}}\sin 2r\theta + \ldots$$

and $\frac{d^{2r}Q}{du^{2r}}\cos r\theta = 0$, and therefore $\cos 2r\theta = 0$,

since $\frac{d^{2r}Q}{du^{2r}}$ cannot be zero when $\frac{d^{2r}P}{du^{2r}} = 0$.

Hence $2r\theta = n\pi + \frac{\pi}{2}$, where n has all values $0, 1, 2, \ldots 2r-1$. Then

$$\lfloor 2r\,(P'-P) = (-)^{n+1}\rho^{2r}\frac{d^{2r}Q}{du^{2r}} + \ldots,$$

and therefore $P'-P$ is alternately positive and negative as n has successively the values $0, 1, 2, \ldots r-1$.

Next, let $\frac{d^{2r}Q}{du^{2r}} = 0$. Then

$$\lfloor 2r\,(P'-P) = \rho^{2r}\frac{d^{2r}P}{du^{2r}}\cos 2r\theta + \ldots,$$

and $\frac{d^{2r}P}{du^{2r}}\sin 2r\theta = 0$, whence $2r\theta = n\pi$, where n has the values $0, 1, 2, \ldots 2r-1$. Thus

$$\lfloor 2r\,(P'-P) = (-)^{n}\rho^{2r}\frac{d^{2r}P}{du^{2r}} + \ldots,$$

which is alternately positive and negative for the successive values of n.

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Next let $\frac{d^{2r}P}{du^{2r}}$ and $\frac{d^{2r}Q}{du^{2r}}$ be neither zero. Then

$$\underline{|2r}\,(P'-P)=\rho^{2r}\frac{\cos 2r\theta}{\frac{d^{2r}P}{du^{2r}}}\left\{\left(\frac{d^{2r}P}{du^{2r}}\right)^2+\left(\frac{d^{2r}Q}{du^{2r}}\right)^2\right\}+\ldots,$$

and, if α be a value of θ from the equation,

$$\tan 2r\theta=-\frac{\frac{d^{2r}Q}{du^{2r}}}{\frac{d^{2r}P}{du^{2r}}},$$

then $2r\theta=n\pi+\alpha$, where n has the values $0, 1, 2, \ldots, 2r-1$, and therefore

$$\underline{|2r}\,(P'-P)=(-)^n.\rho^{2r}\cos\alpha\,\frac{\left(\frac{d^{2r}P}{du^{2r}}\right)^2+\left(\frac{d^{2r}Q}{du^{2r}}\right)^2}{\frac{d^{2r}P}{du^{2r}}}+\ldots,$$

and therefore $P'-P$ has alternately $+$ and $-$ signs as n receives its successive values.

This shews that a series of branches pass through the point in question on the surface $w=P$, the convexities of which are alternately up and down, the value of P being a maximum in regard to each of the former and a minimum in regard to each of the latter branches. The planes of these branches in the vicinity of the point, by what has been written above respecting rhizic curves, divide cylindrical space equiangularly.

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ON A PROBLEM OF ELIMINATION.

By Prof. CAYLEY.

I WRITE

$$P=(a, \ldots)(x, y, z)^k, \quad Q=(a', \ldots)(x, y, z)^k,$$
$$U=(a, \ldots)(x, y, z)^m, \quad V=(b, \ldots)(x, y, z)^n,$$

and I seek for the form of the relation between the coefficients $(a, \ldots)$, $(a', \ldots)$, $(a, \ldots)$, $(b, \ldots)$, in order that there may exist in the pencil

$$\lambda P+\mu Q=0$$

a curve passing through *two* of the intersections of the curves $U=0$, $V=0$.

The ratio $\lambda : \mu$ may be determined so as that the curve $\lambda P+\mu Q=0$ shall pass through one of the intersections of the curves $U=0$, $V=0$; or, what is the same thing, so as that the three curves shall have a common point; the condition for this is

$$\text{Rest. } (\lambda P+\mu Q, U, V)=0,$$

a condition of the form

$$(\lambda a+\mu a', \ldots)^{mn} (a, \ldots)^{kn} (b, \ldots)^{km}=0;$$

or, what is the same thing,

$$(a, \ldots, a', \ldots)^{mn} (a, \ldots)^{kn} (b, \ldots)^{km} (\lambda, \mu)^{mn}=0,$$

which, for shortness, may be written

$$(A, \ldots)(\lambda, \mu)^{mn}=0.$$

Suppose this equation has equal roots, then we have

$$\text{Disct. Rest. } (\lambda P+\mu Q, U, V)=0,$$

the discriminant being taken in regard to λ, μ. This is of the form

$$(A, \ldots)^{2(mn-1)}=0;$$

that is,

$$(a, \ldots, a', \ldots)^{2mn(mn-1)} (a, \ldots)^{2kn(mn-1)} (b, \ldots)^{2km(mn-1)}=0.$$

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It is moreover clear that the nilfactum is a combinant of the functions P, Q; and the form of the equation is therefore

$$\left(\left\| \begin{matrix} \alpha, \beta, \cdots \\ \alpha', \beta', \cdots \end{matrix} \right\|\right)^{mn(mn-1)} (a, \ldots)^{2kn(mn-1)} (b, \ldots)^{2km(mn-1)} = 0.$$

Now the equation in question will be satisfied, 1°. if the curves $U=0$, $V=0$ touch each other; let the condition for this be $\nabla = 0$. 2°. If there exists a curve $\lambda P + \mu Q = 0$ passing through two of the intersections of the curves $U=0$, $V=0$; let the condition be $\Omega = 0$. There is reason to think that the equation contains the factor Ω^2, and that the form thereof is $\Omega^2 \nabla = 0$.

Assuming that this is so, and observing that ∇, the osculant or discriminant of the functions U, V, is of the form

$$\nabla = (a, \ldots)^{m(m+2n-2)} (b, \ldots)^{m(m+2n-2)},$$

we have

$$\Omega^2 = \left(\left\| \begin{matrix} \alpha, \beta, \cdots \\ \alpha', \beta', \end{matrix} \right\|\right)^{mn(mn-1)} (a, \ldots)^{kn(n-1)(2m-1)+(k-1)n(n+2m-2)}, \quad (b, \ldots)^{km(m-1)(2n-1)+(k-1)m(m+2n-2)},$$

and consequently

$$\Omega = \left(\left\| \begin{matrix} \alpha, \beta, \cdots \\ \alpha', \beta', \end{matrix} \right\|\right)^{\frac{1}{2}mn(mn-1)} \cdot (a, \ldots)^{\frac{1}{2}n(n-1)k(2m-1)+\frac{1}{2}(k-1)n(n+2m-2)} \cdot (b, \ldots)^{\frac{1}{2}m(m-1)k(2n-1)+\frac{1}{2}(k-1)m(m+2n-2)},$$

which is the solution of the proposed question. Suppose for instance $n=1$, then

$$\Omega = \left(\left\| \begin{matrix} \alpha, \beta, \cdots \\ \alpha', \beta', \end{matrix} \right\|\right)^{\frac{1}{2}m(m-1)} (a, \ldots)^{(k-1)(m-1)} (b, \ldots)^{\frac{1}{2}m(m-1)k+\frac{1}{2}(k-1)(m-1)}.$$

If moreover $k=1$, then

$$\Omega = \left(\left\| \begin{matrix} \alpha, \beta, \cdots \\ \alpha', \beta', \end{matrix} \right\|\right)^{\frac{1}{2}m(m-1)} (b, \ldots)^{\frac{1}{2}m(m-1)};$$

this is right, for writing $P = \alpha x + \beta y + \gamma z$, $Q = \alpha' x + \beta' y + \gamma' z$, $V = bx + b'y + b''z$, then if two of the intersections of the curve $U=0$, with the line $V=0$ lie in a line with the point $P=0$, $Q=0$, then the point in question, that is the point $(\beta\gamma' - \beta'\gamma,\ \gamma\alpha' - \gamma'\alpha,\ \alpha\beta' - \alpha'\beta)$, must lie in the line $V=0$; and the condition reduces itself to

$$\{(\beta\gamma' - \beta'\gamma,\ \gamma\alpha' - \gamma'\alpha,\ \alpha\beta' - \alpha'\beta)(b, b', b'')\}^{\frac{1}{2}m(m-1)} = 0,$$

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where the index $\frac{1}{2}m(m-1)$ is accounted for as denoting the number of pairs of points out of the m intersections of the curve $U=0$ with the line $V=0$.

If in general $k=1$, then writing as before $P=\alpha x+\beta y+\gamma z$, $Q=\alpha' x+\beta' y+\gamma' z$, we have

$$\Omega=(\beta\gamma'-\beta'\gamma, \ldots)^{\frac{1}{2}mn(mn-1)}\,(a, \ldots)^{\frac{1}{2}m(n-1)(2n-1)}\,(b, \ldots)^{\frac{1}{2}n(m-1)(2m-1)},$$

where $\Omega=0$ is the condition in order that the point $(\beta\gamma'-\beta'\gamma, \ldots)$ may lie *in lineâ* with two of the intersections of the curves $U=0$, $V=0$. Or writing (X, Y, Z) for the coordinates of the given point, the condition is

$$\Omega=(a, \ldots)^{\frac{1}{2}n(n-1)(2n-1)}\,(b, \ldots)^{\frac{1}{2}m(m-1)(2m-1)}\,(X, Y, Z)^{\frac{1}{2}mn(mn-1)}=0.$$

I have found that if

$$U=(a, \ldots)(x, y, z)^m, \quad V=(b, \ldots)(x, y, z)^n,$$

$$W=(c, \ldots)(x, y, z)^p, \quad T=(d, \ldots)(x, y, z)^q,$$

the condition in order that the point (X, Y, Z) may lie *in lineâ* with one of the intersections of the curves $U=0$, $V=0$, and one of the intersections of the curves $W=0$, $T=0$ is

$$\Omega=(a, \ldots)^{npq}\,(b, \ldots)^{mpq}\,(c, \ldots)^{mnq}\,(d, \ldots)^{mnp}\,(X, Y, Z)^{mnpq}=0.$$

Supposing that the curves $W=0$, $T=0$ become identical with the curves $U=0$, $V=0$ respectively, this becomes

$$\Omega=(a, \ldots)^{n^2.2m}\,(b, \ldots)^{m^2.2n}\,(X, Y, Z)^{mn.mn}=0,$$

and the variation from the correct form given above is what might have been expected.

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EQUILIBRIUM OF ONE HEAVY ROUGH BODY ON ANOTHER.

By E. J. Routh.

A PERFECTLY rough heavy body rests in equilibrium on a fixed surface, it is required to determine whether the equilibrium is stable or unstable. We shall suppose the body to be displaced in a plane of symmetry so that the problem may be considered to be one of two dimensions. The case in which the small displacement is any whatever and the fixed surface a horizontal plane is considered in the author's *Rigid Dynamics*, p. 394.

Let the body be displaced into any position and let G be the centre of gravity, A the point of contact. Draw the common normal $C'AC$. Let ϕ be the angle AG makes with AC, ψ' the angle $C'AC$ makes with the vertical. Draw AV vertical and let θ be the angle VAG.

Let A' be any point on the upper surface in the neighbourhood of A, join GA' and let the normal at A' meet AC in C. Let $AG=r$, $AA'=ds$, and let ρ be the radius of curvature of AA' at A. Now

$$G\hat{A}C+\hat{A}GA'=\text{exterior angle } AnA'$$

$$=G\hat{A}'C+A\hat{C}A';$$

therefore
$$\phi+\frac{ds\cos\phi}{r}=\phi+d\phi+\frac{ds}{\rho};$$

therefore
$$\frac{d\phi}{ds}=\frac{\cos\phi}{r}-\frac{1}{\rho}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let ρ' (fig. 13) be the radius of curvature of the lower surface at A. We have

$$\theta=\psi'-\phi;$$

therefore
$$\frac{d\theta}{ds}=\frac{d\psi'}{ds}-\frac{d\phi}{ds};$$

therefore
$$=\frac{1}{\rho'}+\frac{1}{\rho}-\frac{\cos\phi}{r}\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

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Lastly, we have

$$\frac{dr}{ds} = \sin\phi \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

These three equations will supply all the conditions of stability.

In equilibrium the centre of gravity of the body must be vertically over the point of support. Hence $\theta = 0$. In any other position of the body the value of θ is given by the series

$$\frac{d\theta}{ds}\delta s + \frac{d^2\theta}{ds^2}\frac{\delta s^2}{1.2} + \&c.$$

If in this series the first coefficient which does not vanish be positive and of an odd order, it is clear that the line AG will move to the same side of the vertical as that to which the body is moved. The equilibrium will therefore be unstable for displacements on either side of the position of equilibrium. If the coefficient be negative, the equilibrium will be stable. On the other hand, if the term be of an even order it will not change sign with δs, the equilibrium will therefore be stable for a displacement on one side and unstable for one on the other side.

The first differential coefficient is given by (2). The second may be found by differentiating (2) and substituting for $\frac{d\phi}{ds}$ and $\frac{dr}{ds}$ from (1) and (3). The third differential coefficient may be found by repeating this process. In this manner we may find any differential coefficient which may be required.

Case I. Suppose the body such that $\frac{d\theta}{ds}$ is not zero in the position of equilibrium. The condition of stability is that

$$\frac{1}{\rho} + \frac{1}{\rho'} - \frac{\cos\phi}{r}$$

is negative. On the common normal at A measure a length $AS = z$, where

$$\frac{1}{z} = \frac{1}{\rho} + \frac{1}{\rho'},$$

and on AS (fig. 14) as diameter describe a circle, then the equilibrium is stable or unstable according as the centre of gravity lies within or without this circle. Another demonstration of this theorem is given in the author's *Rigid*

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Dynamics, p. 278. For the sake of reference we shall call this circle the circle of stability.

In this reasoning it has been assumed that the centre of gravity is *above* the point of support so that r is positive. If the centre of gravity be below the point of contact of the two surfaces, the condition of stability is that the first differential coefficient of θ, which does not vanish in the position of equilibrium, shall be of an odd order and *positive*. This leads to the condition that

$$\frac{1}{\rho} + \frac{1}{\rho'} + \frac{\cos\phi}{r}$$

shall be positive, where r is the distance of the centre of gravity from the point of contact taken as positive. This expression is always positive, so that the equilibrium is always stable. For, suppose ρ' to be negative, so that the lower body has its concavity upwards, then the upper body must also have its concavity upwards, and its radius of curvature ρ must be numerically less than ρ', thus $\frac{1}{\rho} + \frac{1}{\rho'}$ is positive. In the same way, if ρ be negative we arrive at a similar result.

Case II. Suppose the system to be such that, in the position of equilibrium, the centre of gravity lies on the circumference of the circle of stability. The equilibrium is then neutral to a first approximation. Following our rule we must consider the sign of $\frac{d^2\theta}{ds^2}$. Differentiating (2), we get

$$\frac{d^2\theta}{ds^2} = \frac{d}{ds}\left(\frac{1}{z}\right) + \frac{2\sin\phi\cos\phi}{r^2} - \frac{\sin\phi}{r\rho}.$$

Substituting for $\frac{\cos\phi}{r}$ its value $\frac{1}{\rho} + \frac{1}{\rho'}$, this becomes

$$\frac{d^2\theta}{ds^2} = \frac{d}{ds}\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) + \tan\phi\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)\left(\frac{1}{\rho} + \frac{2}{\rho'}\right);$$

unless this vanishes the equilibrium will be stable for displacements on one side and unstable for displacements on the other side of the position of equilibrium.

If the surfaces in contact be symmetrical about CAC', we have $\frac{d}{ds}\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) = 0$, and the expression cannot vanish unless

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either $\phi = 0$ or $\rho' = -2\rho$. The first of these requires that the moving body should be on the summit of the fixed surface; the second requires that the two surfaces should have their curvatures at the point of contact turned the same way, and the radius of curvature of the lower double that of the other.

If the lower surface be a rough inclined plane, we have ρ' infinite. We have then

$$\frac{d^2\theta}{ds^2} = -\frac{1}{\rho^2}\left(\frac{d\rho}{ds} - \tan\phi\right).$$

If we describe the conic of closest contact* to the upper surface at A (fig. 15), we know that its centre lies on a straight line AO, making with AZ an angle α given by $\tan\alpha = \frac{1}{3}\frac{d\rho}{ds}$. It is therefore necessary for stability that the centre of this conic should lie on the straight line AO, where $\tan ZAO = \frac{1}{3}\tan GAZ$.

Case III. Suppose the system such that $\frac{d^2\theta}{ds^2}$ is also zero in the position of equilibrium. Then taking

$$\frac{d^3\theta}{ds^3} = \frac{d}{ds}\left(\frac{1}{z}\right) + \frac{2\sin\phi\cos\phi}{r^2} - \frac{\sin\phi}{r\rho},$$

we easily find

$$\frac{d^3\theta}{ds^3} = \frac{d^2}{ds^2}\left(\frac{1}{z}\right) - \frac{\sin\phi}{r}\frac{d}{ds}\left(\frac{1}{\rho}\right) + \left(\frac{2\cos 2\phi}{r^2} - \frac{\cos\phi}{r\rho}\right)\left(\frac{\cos\phi}{r} - \frac{1}{\rho}\right) + \left(\frac{4\sin\phi\cos\phi}{-r^3} + \frac{\sin\phi}{r^2\rho}\right)\sin\phi,$$

* Two propositions are wanted in this paper concerning the conic of closest contact at any point A of a curve.

1. The locus of the centres of all conics having contact of the third order with the curve at A is a straight line AG making an angle ϕ with the normal AC, and in front of it, where $\tan\phi = \frac{1}{3}\frac{d\rho}{ds}$.

2. If G be the centre of the conic having contact of the fourth order, then

$$\frac{\cos\phi}{AG} = \frac{1}{\rho} - \frac{d\phi}{ds}.$$

The first property follows from conics. Considering ϕ to be the angle which the radius vector from the centre makes with the normal and ρ the radius

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the first line containing the partial differential coefficient with respect to ρ and ρ', the second with respect to ϕ, the third with respect to r. Substituting for r its value from

$$\frac{\cos\phi}{r} = \frac{1}{\rho} + \frac{1}{\rho'},$$

we find

$$\frac{d^3\theta}{ds^3} = \frac{d^2}{ds^2}\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) + \left(\frac{1}{\rho} + \frac{1}{\rho'}\right)\left(\frac{1}{\rho} + \frac{2}{\rho'}\right)\frac{1}{\rho'}$$
$$- \tan\phi\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)\frac{d}{ds}\left(\frac{1}{\rho}\right) - 3\tan^2\phi\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)^2\left(\frac{1}{\rho} + \frac{2}{\rho'}\right).$$

The equilibrium is stable or unstable according as this expression is negative or positive.

Let us suppose the upper body when in equilibrium to be on the summit of the other, then $\phi = 0$ and the expression is reduced to the two first terms. If the surfaces in contact be both spherical, the condition of stability is that the second term shall be negative. This is impossible unless ρ' be negative. Putting $\rho' = -R$, the condition of stability is that R shall be greater than 2ρ. In this case we have

$$z - 2\rho = -\frac{\rho}{R-\rho}(R - 2\rho),$$

so that the condition of stability is that the circle of stability must lie within the circle of curvature of the upper surface at the point of contact. The result is that if a convex spherical surface rest on the summit of a fixed convex spherical surface, the centre of gravity being at such a height above the point of contact that the equilibrium is neutral to a first approximation, then the equilibrium is really unstable. But if the lower surface have its concavity upwards, the equilibrium is stable or unstable according as its radius of curvature is greater or less than twice that of the upper surface. In the exceptional case in which $R = 2\rho$, we have $\frac{d^3\theta}{ds^3} = 0$. But it is unnecessary to examine the higher dif-

of curvature of the conic, the property may be easily proved for any conic.

The second property follows from equation (1) in the text of this paper. For, s being now measured in the opposite direction, we must replace $\frac{d\phi}{ds}$ by $-\frac{d\phi}{ds}$, and r being AG, we have by simple transposition the equation required.

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ferential coefficients of θ, for the system will now consist of a sphere rolling inside a sphere of double radius with the centre of gravity of the moving body in the circumference. In this case we know the path of the centre of gravity is a *horizontal* straight line, so that the equilibrium is really neutral.

If the surfaces in contact are not spherical, we must find the values of $\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)$ and $\frac{d^2}{ds^2}\left(\frac{1}{\rho'}\right)$ in the position of equilibrium from the known properties of the surfaces. This may often be conveniently effected by expanding $\frac{1}{\rho}$ and $\frac{1}{\rho'}$ in powers of s, neglecting all powers of s beyond the second. Supposing one body still on the summit of the other, we see that if the convexities of the bodies are turned to each other, it is necessary, though not a sufficient condition of stability, that $\frac{d}{ds}\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)=0$ and $\frac{d^2}{ds^2}\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)$ be negative. Hence as the upper body rolls on the lower the radius of the circle of stability must be a minimum in the position of equilibrium.

If the lower surface be an inclined plane, we have ρ' infinite, and the conditions of stability are

$$\frac{d^2\theta}{ds^2}=\frac{1}{\rho^2}\left(\tan\phi-\frac{d\rho}{ds}\right)=0,$$

and

$$\frac{d^3\theta}{ds^3}=\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)-\frac{\tan\phi}{\rho}\frac{d}{ds}\left(\frac{1}{\rho}\right)-3\frac{\tan^2\phi}{\rho^3}$$

$$=\text{a negative quantity.}$$

Substituting for $\tan\phi$ in the latter expression from the first, it reduces to

$$\frac{d^3\theta}{ds^3}=-\frac{1}{\rho^2}\frac{d^2\rho}{ds^2},$$

the equilibrium is therefore stable or unstable according as $\frac{d^2\rho}{ds^2}$ is positive or negative. We may put this into a geometrical form if we replace the section of the moving body by its conic of closest contact. Let O be the centre, $AO=r'$ and the angle $ZAO=\alpha$. Then we know that

$$\frac{d^2\rho}{ds^2}=\frac{3\left(\frac{1}{\rho}-\frac{\cos\alpha}{r'}\right)}{\cos^3\alpha}.$$

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Hence the equilibrium is stable or unstable according as the centre of the conic of closest contact lies without or within the circle of stability.

If the moving body be acted on by a centre of force varying as the distance instead of by gravity, we may determine the nature of the equilibrium in the same way. Let O (fig. 16) be the centre of force, ξ = the angle AGO, $x = OG$, and let the rest of the notation be as before. If the upper body roll until the point A' come into contact with the fixed surface, the angle rolled through is clearly $d\omega = \frac{ds}{z}$, where $\frac{1}{z} = \frac{1}{\rho} + \frac{1}{\rho'}$. Therefore if G come into the position G', we have $GG' = rd\omega$, and therefore

$$dx = -rd\omega \sin\xi;$$

therefore

$$\frac{dx}{ds} = -\frac{r}{z}\sin\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Again, if χ be the angle OG makes with the vertical,

$$d\chi = \frac{rd\omega\cos\xi}{x},$$

and

$$\xi = \theta - \chi;$$

therefore

$$\frac{d\xi}{ds} = \frac{d\theta}{ds} - \frac{r\cos\xi}{xz}$$

$$= \frac{1}{z} + \frac{\cos\phi}{r} - \frac{r\cos\xi}{xz} \quad \ldots\ldots\ldots (5).$$

In equilibrium $\xi = 0$, and when slightly displaced, we have

$$\xi = \frac{d\xi}{ds}\delta s + \frac{d^2\xi}{ds^2}\frac{\delta s^2}{1.2} + \ldots .$$

The values of the successive differential coefficients of ξ may be found by differentiating (5) and substituting from (4), (1), and (3) the values there given of $\frac{dx}{ds}$, $\frac{d\phi}{ds}$, and $\frac{dr}{ds}$. The condition of stability is that the first differential coefficient which does not vanish must be of an odd order and negative. Suppose that $\frac{d\xi}{ds}$ does not vanish, then the equilibrium is stable if $\frac{1}{z} - \frac{r}{xz} - \frac{\cos\phi}{r}$ be negative. Put $AO = x'$ so that

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$x = x' + r$, and let us suppose both x' and r to be positive. Then the equilibrium is stable or unstable according as

$$\frac{1}{z} - \left(\frac{1}{x'} + \frac{1}{r}\right)\cos\phi$$

is negative or positive. We may express this result geometrically in the following manner. Take a point G_1 in AG so that

$$\frac{1}{AG_1} = \frac{1}{AG} + \frac{1}{AO},$$

G and O being on opposite sides of A, then the equilibrium will be stable or unstable according as G_1 lies within or without the circle of stability.

MOMENT OF INERTIA OF A QUADRILATERAL.

By E. J. Routh.

If α, β, γ, δ, ε be the ordinates of the angular points A, B, C, D and of the intersection E of the diagonals of a quadrilateral area, then the ordinate of its centre of gravity is $\frac{1}{4}(\alpha+\beta+\gamma+\delta-\varepsilon)$. This is a very convenient formula for finding the centre of gravity of a quadrilateral area. The proof is very easy and follows at once from a different construction given in the sixth volume of the *Quarterly Journal*.

The moment of inertia of a quadrilateral about any straight line in its plane may be proved to be

$$\frac{M}{6}\{\alpha^2+\beta^2+\gamma^2+\delta^2+\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta - \varepsilon(\alpha+\beta+\gamma+\delta)\}$$

in the following manner:

Let Δ, Δ' be the areas of the two triangles ADC, ABC, then if M be the area of the whole quadrilateral, we have

$$\left.\begin{aligned}\Delta &= M\frac{DE}{DB} = M\frac{\delta-\varepsilon}{\delta-\beta}\\ \Delta' &= M\frac{BE}{BD} = M\frac{\varepsilon-\beta}{\delta-\beta}\end{aligned}\right\}.$$

We know that the moment of inertia of a triangle about any straight line is the same as that of three particles each one-

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third the mass of the triangle placed at the middle points of the sides. Hence the moment of inertia of the quadrilateral about any straight line in its plane is

$$\frac{\Delta}{3}\left\{\left(\frac{\alpha+\delta}{2}\right)^2+\left(\frac{\delta+\gamma}{2}\right)^2+\left(\frac{\gamma+\alpha}{2}\right)^2\right\}$$

$$+\frac{\Delta'}{3}\left\{\left(\frac{\alpha+\beta}{2}\right)^2+\left(\frac{\beta+\gamma}{2}\right)^2+\left(\frac{\gamma+\alpha}{2}\right)^2\right\}.$$

Substituting for Δ and Δ' their values in terms of M, we get, after an easy reduction, the result given above.

Let four equal particles, the mass of each being $\frac{M}{12}$, be placed one at each angular point, and let a negative mass $=-\frac{M}{12}$ be placed at the intersection of the diagonals, also let a mass $\frac{3}{4}M$ be placed at the centre of gravity of the area of the quadrilateral. Then the moments of inertia of the area of the quadrilateral is equal to that of these six particles about any straight line in the plane of the area. For the moment of inertia of these six particles is clearly

$$\frac{M}{12}(\alpha^2+\beta^2+\gamma^2+\delta^2-\varepsilon^2)+\frac{3}{4}M\left(\frac{\alpha+\beta+\gamma+\delta-\varepsilon}{3}\right)^2,$$

and this is easily seen to be the same as that of the quadrilateral area.

These six particles treated as one system and the quadrilateral have a common centre of gravity. Also, they have the same moment of inertia about any straight line in the plane of the area. Hence, by the same reasoning as in the author's *Rigid Dynamics*, Art. 17, the six particles and the quadrilateral have the same moments of inertia about any straight line whatever, and the same products of inertia about any two perpendicular straight lines.

As an example of the application of this rule, let us find the depth of the centre of pressure of any vertical quadrilateral area. In any plane figure intersecting the effective surface of the fluid in the axis of x, the coordinates of the centre of pressure are given by

$$Y=\frac{\text{moment of inertia about } Ox}{\text{moment of area about } Ox},$$

$$X=\frac{\text{product of inertia about } Ox,\ Oy}{\text{moment of area about } Ox}.$$

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We have therefore

$$Y = \frac{1}{4}\,\frac{\alpha^2+\beta^2+\gamma^2+\delta^2-\varepsilon^2+(\alpha+\beta+\gamma+\delta-\varepsilon)^2}{\alpha+\beta+\gamma+\delta-\varepsilon},$$

where $\alpha, \beta, \gamma, \delta, \varepsilon$ are the depths of the angular points and of the intersection of the diagonals. The value of X may be written down in the same way.

ON THE QUARTIC SURFACES $(*\nabla U, V, W)^2 = 0$.

(Continued from Vol. XI, p. 25).

By Prof. CAYLEY.

THE general Torus, or surface generated by the rotation of a conic about a fixed axis anywise situate, has been investigated by M. De La Gournerie, *Jour. de l'Ecole Polyt.*, t. XXIII. (1863) pp. 1 to 74. The surface is one of the fourth order, having a nodal circle; and with its equation of the form $V^2 - UW = 0$, consequently of the form in question. The leading points of the theory are as follows:

Consider (fig. 17) the plane of the conic in any particular position thereof; let this meet the axis of rotation OO' in the point M, and let the projection of OO' on the plane of the conic be MN. Take P any point of the conic; draw PQ in the plane of the conic, perpendicular to MN, and QR perpendicular to OO', and join PR: the point P of the conic describes a circle radius RP, $= \sqrt{(RQ^2 + QP^2)}$. Hence if $\angle OMN = \alpha$, and if $MQ = \mathrm{x}$, $QP = \mathrm{y}$ are the coordinates in the plane of the conic of the point P; and if the coordinates x, y, z are measured, z upwards from M in the direction MO, and x, y in the plane at right angles to the axis OO': we have

$$z = \mathrm{x}\cos\alpha, \quad \sqrt{(x^2+y^2)} = \sqrt{(\mathrm{x}^2\sin^2\alpha + \mathrm{y}^2)};$$

or, what is the same thing,

$$\mathrm{x} = z\sec\alpha, \quad \mathrm{y} = \sqrt{(x^2+y^2-z^2\tan^2\alpha)}.$$

Hence the equation of the conic being $F(\mathrm{x}, \mathrm{y}) = 0$, that of the torus is $F\{z\sec\alpha, \sqrt{(x^2+y^2-z^2\tan^2\alpha)}\} = 0$.

Thus taking the equation of the conic to be

$$(a, b, c, f, g, h\nabla\mathrm{x}, \mathrm{y}, 1)^2 = 0;$$

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or, as this may be written,

$$(ax^2 + 2gx + c + by^2)^2 = 4y^2 (hx + f)^2,$$

we have at once the equation of the torus in the form

$$\{az^2 \sec^2\alpha + 2gz \sec\alpha + c + b(x^2 + y^2 - z^2 \tan^2\alpha)\}^2$$
$$= 4(x^2 + y^2 - z^2 \tan^2\alpha)(hz \sec\alpha + f)^2,$$

which is of the form $V^2 - 4UW = 0$; or, as it is better to write it $V^2 - 4UL^2 = 0$, where

$$V = az^2 \sec^2\alpha + 2gz \sec\alpha + c + b(x^2 + y^2 - z^2 \tan^2\alpha),$$
$$U = x^2 + y^2 - z^2 \tan^2\alpha,$$
$$L = hz \sec\alpha + f, \quad W = L^2.$$

There is thus a nodal circle $V = 0$, $L = 0$, that is

$$z = -\frac{f}{h} \cos\alpha,$$

$$bh^2(x^2 + y^2) - bf^2 \sin^2\alpha + af^2 - 2gfh + ch^2 = 0.$$

But the origin of this nodal circle is better seen geometrically. For observe that the radius of the circle described by the point P of the conic depends only on the *square* of the ordinate PQ: hence if we have on the conic two points S, S' situate symmetrically in regard to the line MN, these points S, S' will describe one and the same circle, which will be a nodal circle on the surface. And there is in fact one such pair of points S, S'; for (see fig. 18) considering in the plane of the conic the equal conic situate symmetrically thereto on the other side of the line MN, the two conics intersect in two points T, T' (real or imaginary) on the line MN, and in two other points S, S' (real or imaginary) situate symmetrically in regard to MN; we have thus the required pair of points which generate the nodal circle.

A meridian section of the torus (or section through the axis OO') is a quartic curve symmetrical in regard to this axis, and having two (real or imaginary) nodes the intersections of the plane by the nodal circle: see fig. 19, which shows the section for the surface generated by a conic such as in fig. 18. The quartic curve has 8 double tangents, 2 of them at right angles to the axis OO', the remaining 6 forming 3 pairs of tangents situate symmetrically in regard to this axis; so that attending only to one tangent of each pair, we may say that there are 3 oblique bitangents; one of these is the line TT', the section of the torus by a plane through

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this line at right angles to the plane of the meridian section, being in fact the two conics of fig. 18, either of which by its rotation about OO' generates the torus. But taking either of the other two oblique bitangents, the section by a plane through the bitangent at right angles to the meridian plane is in like manner a pair of conics situate symmetrically in regard to the bitangent, and such that either of them by its rotation about the axis OO' generates the torus. It thus appears that the same torus may be generated in three different ways by the rotation of a conic about the axis OO'.

In the particular case where the plane of the conic passes through the axis, the meridian section consists it is clear of two symmetrically situate conics, intersecting the axis in the points T, T'', which are nodes of the surface, the surface having as before a nodal circle generated by the rotation of the two symmetrically situate intersections S, S' of the two conics. The equation is included under the foregoing from, but it is at once obtained from that of the conic,

$$(a\mathrm{x}^2 + 2g\mathrm{x} + c + b\mathrm{y}^2)^2 = 4\mathrm{y}^2 (h\mathrm{x} + f)^2,$$

by writing therein z for x and $\sqrt{(x^2 + y^2)}$ for y; viz. the equation of the torus here is

$$\{az^2 + 2gz + c + b(x^2 + y^2)\}^2 = 4(x^2 + y^2)(hz + f)^2,$$

and the two nodes thus are $x = 0$, $y = 0$, $az^2 + 2gz + c = 0$.

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ON AXES OF LUG AND AXES OF KICK.

By WILLIAM WALTON, M.A., Fellow of Trinity Hall.

WHEN a rigid body is revolving spontaneously about a fixed point, the pressure on the point is generally inclined to the instantaneous axis at an acute angle. The object of this article is to investigate the circumstances of two exceptional cases: (1) when the direction of the pressure on the point coincides with the instantaneous axis, (2) when it is at right angles to the instantaneous axis. For want of better terms, I propose to call the instantaneous axes in the two cases respectively *axes of lug* and *axes of kick*. The notation adopted in two articles on the spontaneous rotation of a rigid body, printed in the First Number of this Volume of the *Quarterly Journal*, will be adhered to in the following essay.

Axis of Lug.

The components of the reaction of the fixed point are given by the equations

$$\frac{X}{M} = (b+c-a).\frac{\omega_1}{bc}.(b\beta\omega_2 + c\gamma\omega_3) - \alpha(\omega_2^2 + \omega_3^2),$$

$$\frac{Y}{M} = (c+a-b).\frac{\omega_2}{ca}.(c\gamma\omega_3 + a\alpha\omega_1) - \beta(\omega_3^2 + \omega_1^2),$$

$$\frac{Z}{M} = (a+b-c).\frac{\omega_3}{ab}.(a\alpha\omega_1 + b\beta\omega_2) - \gamma(\omega_1^2 + \omega_2^2).$$

Imagine this reaction to coincide in direction with the instantaneous axis: then

$$\frac{b+c-a}{bc}(b\beta\omega_2 + c\gamma\omega_3) - \frac{\alpha}{\omega_1}(\omega_2^2 + \omega_3^2)$$

$$= \frac{c+a-b}{ca}(c\gamma\omega_3 + a\alpha\omega_1) - \frac{\beta}{\omega_2}(\omega_3^2 + \omega_1^2)$$

$$= \frac{a+b-c}{ab}(a\alpha\omega_1 + b\beta\omega_2) - \frac{\gamma}{\omega_3}(\omega_1^2 + \omega_2^2).$$

From these three equations we may easily ascertain that

$$\frac{\alpha}{P} = \frac{\beta}{Q} = \frac{\gamma}{R},$$

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where

$$P=\omega_2\omega_3\left(\frac{c+a-b}{a}\cdot\frac{a+b-c}{a}-\frac{b+c-a}{c}\cdot\frac{c+a-b}{a}-\frac{a+b-c}{a}\cdot\frac{b+c-a}{b}\right)$$

$$-\frac{b+c-a}{b}\cdot\frac{\omega_3}{\omega_2}\cdot(\omega_3^2+\omega_1^2)$$

$$-\frac{b+c-a}{c}\cdot\frac{\omega_2}{\omega_3}\cdot(\omega_1^2+\omega_2^2)$$

$$-\frac{\omega_1^2+\omega_2^2}{\omega_3}\cdot\frac{\omega_3^2+\omega_1^2}{\omega_2},$$

analogous expressions belonging to Q and R.

Now the coefficient of $\omega_2\omega_3$, in the expression for P, is easily seen to be equal to

$$\frac{1}{a}\left\{a-b-c-\frac{(b-c)^2}{a}+\frac{(c-a)^2}{b}+\frac{(a-b)^2}{c}\right\}.$$

Also, the remaining terms in the expression for P, with their signs changed, are equal to

$$\frac{c-a}{b}\cdot\frac{\omega_3}{\omega_2}\cdot(\omega_3^2+\omega_1^2)+\frac{b-a}{c}\cdot\frac{\omega_2}{\omega_3}\cdot(\omega_1^2+\omega_2^2)$$

$$+\frac{\omega_3}{\omega_2}\cdot(\omega_3^2+\omega_1^2)+\frac{\omega_2}{\omega_3}\cdot(\omega_1^2+\omega_2^2)+\frac{\omega_1^2+\omega_2^2}{\omega_3}\cdot\frac{\omega_3^2+\omega_1^2}{\omega_2}$$

$$=\frac{c-a}{b}\cdot\frac{\omega_3}{\omega_2}\cdot(\omega^2-\omega_2^2)+\frac{b-a}{c}\cdot\frac{\omega_2}{\omega_3}\cdot(\omega^2-\omega_3^2)$$

$$+\frac{\omega_3}{\omega_2}\cdot(\omega^2-\omega_2^2)+\frac{\omega_2}{\omega_3}\cdot(\omega^2-\omega_3^2)+\frac{\omega^2-\omega_3^2}{\omega_3}\cdot\frac{\omega^2-\omega_2^2}{\omega_2}$$

$$=\frac{\omega^4}{\omega_2\omega_3}+\omega^2\left(\frac{c-a}{b}\cdot\frac{\omega_3}{\omega_2}+\frac{b-a}{c}\cdot\frac{\omega_2}{\omega_3}\right)$$

$$-\omega_2\omega_3\cdot\left(\frac{c-a}{b}+\frac{b-a}{c}+1\right).$$

Thus

$$P\omega_2\omega_3=\frac{\omega_2^2\omega_3^2}{a}\cdot\left\{a-b-c-\frac{(b-c)^2}{a}+\frac{(c-a)^2}{b}+\frac{(a-b)^2}{c}\right\}$$

$$-\omega^4-\omega^2\left(\frac{c-a}{b}\omega_3^2+\frac{b-a}{c}\omega_2^2\right)$$

$$+\omega_2^2\omega_3^2\left(\frac{c-a}{b}+\frac{b-a}{c}+1\right).$$

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Now we sum the coefficients of $\omega_1\omega_2$ in this expression

hence

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Let
$$u = \{f(c+a)-g^2\}\{f(a+b)-g^2\},$$
$$v = \{f(a+b)-g^2\}\{f(b+c)-g^2\},$$
$$w = \{f(b+c)-g^2\}\{f(c+a)-g^2\},$$
$$u' = a\{2bcf+(af-g^2)(b+c-a)\},$$
$$v' = b\{2caf+(bf-g^2)(c+a-b)\},$$
$$w' = c\{2abf+(cf-g^2)(a+b-c)\}:$$

then the system of equations

$$\frac{\alpha}{P} = \frac{\beta}{Q} = \frac{\gamma}{R},$$

is reduced to the system

$$\frac{\omega_1}{a\alpha}(u-u'\omega^2) = \frac{\omega_2}{b\beta}(v-v'\omega^2) = \frac{\omega_3}{c\gamma}(w-w'\omega^2) \ldots\ (1).$$

From the equations (1), combined with the equations

$$\left.\begin{aligned} \omega_1^2 &= \frac{bc}{(c-a)(a-b)}(\lambda_1-\omega^2) \\ \omega_2^2 &= \frac{ca}{(a-b)(b-c)}(\lambda_2-\omega^2) \\ \omega_3^2 &= \frac{ab}{(b-c)(c-a)}(\lambda_3-\omega^2) \end{aligned}\right\} \ldots\ldots\ldots\ (2),$$

we have

$$\left.\begin{aligned} &\frac{b-c}{a^3\alpha^2}(\lambda_1-\omega^2)(u-u'\omega^2)^2 \\ &= \frac{c-a}{b^3\beta^2}(\lambda_2-\omega^2)(v-v'\omega^2)^2 \\ &= \frac{a-b}{c^3\gamma^2}(\lambda_3-\omega^2)(w-w'\omega^2)^2 \end{aligned}\right\} \ldots\ldots\ldots\ (3).$$

Let a, b, c, be in the descending order of magnitude: then, by the equations (2), ω^2 is not $>\lambda_2$, not $<\lambda_1$, and not $<\lambda_3$. Consequently the three members of the system (3) have all the same sign. Hence, whatever value ω^2 may have within the above limits, the system (3) must necessarily be possible if α, β, γ, bear towards each other a suitable proportion. This amounts to the conclusion that axes of lug may exist under certain conditions.

From the equations

$$a\omega_1^2 + b\omega_2^2 + c\omega_3^2 = f,$$
$$a^2\omega_1^2 + b^2\omega_2^2 + c^2\omega_3^2 = g^2,$$
$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega^2,$$

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we at once obtain the equations

$$a(af-g^2)\omega_1^2+b(bf-g^2)\omega_2^2+c(cf-g^2)\omega_3^2=0\ldots(4),$$

and $$(a\omega^2-f)\omega_1^2+(b\omega^2-f)\omega_2^2+(c\omega^2-f)\omega_3^2=0\ldots(5).$$

From (1) and (4) we have

$$\frac{a^2\alpha^2(af-g^2)}{(u-u'\omega^2)^2}+\frac{b^2\beta^2(bf-g^2)}{(v-v'\omega^2)^2}+\frac{c^2\gamma^2(cf-g^2)}{(w-w'\omega^2)^2}=0\ldots(6).$$

Again, from (1) and (5), we have

$$\frac{a^2\alpha^2(a\omega^2-f)}{(u-u'\omega^2)^2}+\frac{b^2\beta^2(b\omega^2-f)}{(v-v'\omega^2)^2}+\frac{c^2\gamma^2(c\omega^2-f)}{(w-w'\omega^2)^2}=0\ldots(7).$$

If the value of ω^2, derived from (6), be substituted in (7), the resulting equation establishes a relation between $\frac{g^2}{f}$ and the quantities $a, b, c, \alpha, \beta, \gamma$. Each value of $\frac{g^2}{f}$ corresponds to one axial cone, *i.e.*, the cone described relatively to the body by the instantaneous axis, the equation to which is

$$a(af-g^2)x^2+b(bf-g^2)y^2+c(cf-g^2)z^2=0.$$

Since the equation (6) is a biquadratic in ω^2, there are four values of ω^2 for each value of $\frac{g^2}{f}$. If we substitute in the equations (1) the several values of ω^2, and replace $\omega_1, \omega_2, \omega_3$, by x, y, z, respectively, we shall have the equations to the corresponding axes of lug. Thus there will be four axes of lug for each possible positive value of $\frac{g^2}{f}$, which corresponds to four positive possible values of ω^2 not greater than λ_2 and not less than the greater of the quantities λ_1, λ_3.

If we eliminate ω^2 between the equations of the system (1), we shall get

$$\frac{a\alpha}{\omega_1}(vw'-wv')+\frac{b\beta}{\omega_2}(wu'-uw')+\frac{c\gamma}{\omega_3}(uv'-vu')=0\ldots(8).$$

Attending to the equations (4) and (8), we see that the axes of lug, when they exist, coincide with the intersections of the two quadric cones

$$a(af-g^2)x^2+b(bf-g^2)y^2+c(cf-g^2)z^2=0,$$

$$\frac{a\alpha}{x}(vw'-wv')+\frac{b\beta}{y}(wu'-uw')+\frac{c\gamma}{z}(uv'-vu')=0.$$

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Let us discuss the particular case when the centre of gravity lies in the plane of the principal axes of greatest and least moments of inertia, that is, when $\beta = 0$.

The equations (6) and (7) become

$$\frac{a^2\alpha^2(af-g^2)}{(u-u'\omega^2)^2} + \frac{c^2\gamma^2(cf-g^2)}{(w-w'\omega^2)^2} = 0 \ldots\ldots\ldots\ldots (9),$$

$$\frac{a^2\alpha^2(a\omega^2-f)}{(u-u'\omega^2)^2} + \frac{c^2\gamma^2(c\omega^2-f)}{(w-w'\omega^2)^2} = 0 \ldots\ldots\ldots (10),$$

and, from (9) and (10),

$$\frac{a(af-g^2)}{a\omega^2-f} = \frac{c(cf-g^2)}{c\omega^2-f},$$

and consequently

$$\omega^2 = \frac{(c+a)f-g^2}{ca}.$$

Substituting this value of ω^2 in (9), and effecting obvious operations, we shall obtain, as the condition for the existence of an axis of lug, the equation

$$c^{\frac{1}{2}}\gamma(g^2-fc)^{\frac{1}{2}}[g^2(a-b)+f\{b(c+a)-a^2\}]$$
$$= a^{\frac{1}{2}}\alpha(fa-g^2)^{\frac{1}{2}}[g^2(c-b)+f\{b(c+a)-c^2\}] \ldots (11).$$

From the systems (1) and (4), putting $\beta = 0$, we have

$$\omega_2 = 0, \quad c(g^2-fc)\omega_3^2 = a(fa-g^2)\omega_1^2,$$

and therefore the equations to the axes of lug, two in number, are

$$y = 0, \quad c(g^2-fc)z^2 = a(fa-g^2)x^2 \ldots\ldots\ldots (12).$$

If we liberate the equation (11) from radicals, we shall obtain the following equation, viz.,

$$g^6\{c\gamma^2(a-b)^2 + a\alpha^2(c-b)^2\}$$
$$+fg^4\{c\gamma^2(a-b)\}(3bc+2ab-2a^2-ca)$$
$$+a\alpha^2(c-b)(3ab+2bc-2c^2-ca)\}$$
$$+f^2g^2\{c\gamma^2(bc+ab-a^2)(3bc+ab-a^2-2ca)$$
$$+a\alpha^2(ab+bc-c^2)(3ab+bc-c^2-2ca)\}$$
$$-f^3\{c^2\gamma^2(bc+ab-a^2)^2 + a^2\alpha^2(bc+ab-c^2)^2\} = 0 \ldots (13).$$

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Since the equation (13) is a cubic in $\frac{g^2}{f}$, it follows that the equation to the axial cone, viz.,

$$a(af-g^2)x^2+b(bf-g^2)y^2+c(cf-g^2)z^2=0$$

may represent three different cones. Thus the axes of lug, defined by the equations (12), are generating lines of three different axial cones, described in the body by the instantaneous axis in accordance with three different initial motions of the body.

Axis of Kick.

Again, the condition for an axis of kick is

$$\omega_1 X+\omega_2 Y+\omega_3 Z=0,$$

which, in virtue of the expressions for X, Y, Z, becomes

$$\frac{\omega_1^2}{bc}\{b\beta(b-a)\omega_2+c\gamma(c-a)\omega_3\}$$

$$+\frac{\omega_2^2}{ca}\{c\gamma(c-b)\omega_3+a\alpha(a-b)\omega_1\}$$

$$+\frac{\omega_3^2}{ab}\{a\alpha(a-c)\omega_1+b\beta(b-c)\omega_2\}=0,$$

and therefore, substituting the values of ω_1^2, ω_2^2, ω_3^2, in terms of ω^2, in the expressions $\frac{\omega_1^2}{bc}$, $\frac{\omega_2^2}{ca}$, $\frac{\omega_3^2}{ab}$, we get

$$\frac{\lambda_1}{(c-a)(a-b)}\{b\beta(b-a)\omega_2+c\gamma(c-a)\omega_3\}$$

$$+\frac{\lambda_2}{(a-b)(b-c)}\{c\gamma(c-b)\omega_3+a\alpha(a-b)\omega_1\}$$

$$+\frac{\lambda_3}{(b-c)(c-a)}\{a\alpha(a-c)\omega_1+b\beta(b-c)\omega_2\}=0.$$

Substituting for λ_1, λ_2, λ_3, their respective values

$$\frac{f(b+c)-g^2}{bc},\quad \frac{f(c+a)-g^2}{ca},\quad \frac{f(a+b)-g^2}{ab},$$

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the equation becomes

$$f\left[\frac{b^2-c^2}{bc}\{b\beta(b-a)\omega_2+c\gamma(c-a)\omega_3\}\right.$$
$$+\frac{c^2-a^2}{ca}\{c\gamma(c-b)\omega_3+a\alpha(a-b)\omega_1\}$$
$$\left.+\frac{a^2-b^2}{ab}\{a\alpha(a-c)\omega_1+b\beta(b-c)\omega_2\}\right]$$
$$-g^2\left[\frac{b-c}{bc}\{b\beta(b-a)\omega_2+c\gamma(c-a)\omega_3\}\right.$$
$$+\frac{c-a}{ca}\{c\gamma(c-b)\omega_3+a\alpha(a-b)\omega_1\}$$
$$\left.+\frac{a-b}{ab}\{a\alpha(a-c)\omega_1+b\beta(b-c)\omega_2\}\right].$$

Now, in the coefficient of f, if K be taken to denote $(b-c)(c-a)(a-b)$, the coefficients of $\alpha\omega_1$, $\beta\omega_2$, $\gamma\omega_3$, are, as may be easily ascertained, equal respectively to

$$\frac{Ka}{bc}, \frac{Kb}{ca}, \frac{Kc}{ab}:$$

and, in the coefficient of $-g^2$, the coefficients of $\alpha\omega_1$, $\beta\omega_2$, $\gamma\omega_3$, as may be easily shewn, are respectively equal to

$$\frac{K}{bc}, \frac{K}{ca}, \frac{K}{ab}.$$

Thus the condition for an axis of kick is reduced to the equation

$$a\alpha(af-g^2)\omega_1+b\beta(bf-g^2)\omega_2+c\gamma(cf-g^2)\omega_3=0.$$

But the quantities ω_1, ω_2, ω_3, are connected together by the equation

$$a(af-g^2)\omega_1^2+b(bf-g^2)\omega_2^2+c(cf-g^2)\omega_3^2=0.$$

From these two results it follows that the intersections, when possible, of the axial cone

$$a(af-g^2)x^2+b(bf-g^2)y^2+c(cf-g^2)z^2=0,$$

with the plane

$$a\alpha(af-g^2)x+b\beta(bf-g^2)y+c\gamma(cf-g^2)z=0,$$

are axes of kick, two in number.

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In order that the coexistence of these two equations may be possible, we must, as may be easily proved, have the condition

$$(af-g^2)(bf-g^2)(cf-g^2)\{a\alpha^2(af-g^2)+b\beta^2(bf-g^2)+c\gamma^2(cf-g^2)\}=\text{or}<0.$$

From this condition it is evident that, when there is an axis of kick, the centre of gravity of the body does not lie within or without the axial cone, when the sign of

$$(af-g^2)(bf-g^2)(cf-g^2)$$

is respectively positive or negative.

If $af-g^2=0$, then the equations to the axis of kick become

$$b(b-a)y^2+c(c-a)z^2=0,$$

$$b\beta(b-a)y+c(c-a)z=0,$$

and therefore $y=0$, $z=0$: thus the axis of kick coincides with the principal axis of greatest moment of inertia.

From the expressions for X, Y, Z, it may be easily seen that, ω_2^2, ω_3^2, being both zero, and ω_1^2 equal to ω^2, when $af=g^2$, the equations to the reaction of the fixed point on the body are

$$x=0,\quad \frac{y}{\beta}=\frac{z}{\gamma}.$$

If $cf-g^2=0$, then it may be shewn in like manner that the axis of kick coincides with the principal axis of least moment of inertia, the equations to the direction of the reaction being

$$z=0,\quad \frac{x}{\alpha}=\frac{y}{\beta}.$$

If $bf-g^2=0$, then the equations to the axis of kick are

$$a(a-b)x^2=c(b-c)z^2,$$

$$a\alpha(a-b)x=c\gamma(b-c)z.$$

Thus the axis of kick lies anywhere in the plane denoted by the second of these equations, a condition for its existence being

$$a\alpha^2(a-b)=c\gamma^2(b-c),$$

which shews that the centre of gravity lies in this plane. In this case therefore the instantaneous axis is an axis of kick during its whole motion in its plane locus.

July 20, 1870.

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ON THE ALGEBRA OF MAGIC SQUARES. No. II.

By JOSEPH HORNER, M.A., Clare College.

TO prove that every odd zero arithmetical progression of $\rho\sigma$ terms, where $\rho = 2r+1$ and $\sigma = 2s+1$, may be separated into σ zero columns of ρ terms each.

This depends on the fact, that the equation

$$R_\sigma x + R_\sigma (s+x) + R_\sigma (s-2x+1) = 0$$

is true for all integer values of x. Giving x the σ values $-s, -s+1, \ldots s-1, s$, we obtain the following, σ columns of zero triplets

$-s$	$-s+1$	$-s+2$	...	$s-2$	$s-1$	s
0	1	2		-3	-2	-1
s	$s-2$	$s-4$		$-s+5$	$-s+3$	$-s+1$,

where each row necessarily consists of the series

$$-s, -s+1, \ldots s-1, s$$

in different orders. There appears to be no other system of zero triplets, which can be formed by taking any odd zero series whatever three times in this manner. For some particular values of s however, there are other possible arrangements not comprised in the general formula above, and that seem to depend on complex laws to which I possess no clue. Thus when $s=3$, we have the symmetrical arrangement

$$\begin{array}{l} \bar{3}\ \bar{2}\ \bar{1}\ 0\ 1\ 2\ 3 \\ 1\ 3\ \bar{2}\ 0\ 2\ \bar{3}\ \bar{1} \\ 2\ \bar{1}\ 3\ 0\ \bar{3}\ 1\ \bar{2}. \end{array}$$

Also when $s=4$, there are the following forms, one symmetrical, the other unsymmetrical,

$$\begin{array}{l} \bar{4}\ \bar{3}\ \bar{2}\ \bar{1}\ 0\ 1\ 2\ 3\ 4 \\ 3\ \bar{1}\ 4\ \bar{2}\ 0\ 2\ \bar{4}\ 1\ \bar{3} \\ 1\ 4\ \bar{2}\ 3\ 0\ \bar{3}\ 2\ \bar{4}\ \bar{1} \end{array} \quad \text{and} \quad \begin{array}{l} \bar{4}\ \bar{3}\ \bar{2}\ \bar{1}\ 0\ 1\ 2\ 3\ 4 \\ 2\ 4\ \bar{2}\ 3\ 0\ \bar{4}\ \bar{3}\ 1\ \bar{1} \\ 2\ \bar{1}\ 4\ \bar{2}\ 0\ 3\ 1\ \bar{4}\ \bar{3}.* \end{array}$$

No such arrangement is possible when $s=2$.

* These two particular cases were suggested to me by William, only surviving child of the late W. G. Horner.

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Having formed the σ zero triplets, we take any pair of the three rows, diminish each term of the one by σ, and increase each term of the other by that quantity. Thus we still have σ zero triplets.

For example,

$-3s-1$	$-3s$	$-3s+1$	$\ldots -s-3$	$-s-2$	$-s-1$
0	1	2	-3	-2	-1
$3s+1$	$3s-1$	$3s-3$	$s+6$	$s+4$	$s+2$.

Above or below these we may write in rows the pairs of arithmetical series

$$\begin{cases} -5s-2, & -5s-1, \ldots -3s-2 \\ 5s+2, & 5s+1, \ldots \quad 3s+2 \end{cases}$$

$$\begin{cases} -7s-3, & -7s-2, \ldots -5s-3 \\ 7s+3, & 7s+2, \ldots \quad 5s+3 \end{cases}$$

$$\vdots$$

$$\begin{cases} -2rs-r-s, & -2rs-r-s+1, \ldots -2rs-r+s \\ 2rs+r+s, & 2rs+r+s-1, \ldots \quad 2rs+r-s, \end{cases}$$

the rows being in any order, and the terms of each arithmetical series in any order, provided the equal contrasignal terms of each pair be in the same column. Thus the problem is solved in various ways.

Otherwise, we take any pair of the original three rows and diminish each term of the one by 2σ or 3σ or ..., and increase each term of the other by that same quantity; then write above or below these, with the licence just mentioned as to rows and terms, the arithmetical progressions

$$\begin{cases} -3s-1, & -3s, \ldots -s-1 \\ 3s+1, & 3s, \ldots \quad s+1, \end{cases}$$

instead of

$$\begin{cases} -5s-2, & -5s-1, \ldots -3s-2 \\ 5s+2, & 5s+1 \qquad 3s+2, \end{cases}$$

or

$$\begin{cases} -7s-3, & -7s-2, \ldots -5s-3 \\ 7s+3, & 7s+2, \ldots \quad 5s+3, \end{cases}$$

or ..., as the case may be. Thus we obtain various other solutions.

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Ex. $r=2,\ s=2$. The original rows are

$$\begin{cases} \bar{2}\ \bar{1}\ 0\ 1\ 2 \\ 1\ 2\ \bar{2}\ \bar{1}\ 0 \\ 1\ \bar{1}\ 2\ 0\ \bar{2}. \end{cases}$$

Hence

$$\begin{cases} \bar{7}\ \bar{6}\ \bar{5}\ \bar{4}\ \bar{3} \\ 1\ 2\ \bar{2}\ \bar{1}\ 0 \\ 6\ 4\ 7\ 5\ 3, \end{cases}$$

and

$$\begin{cases} \overline{12}\ \overline{11}\ \overline{10}\ \bar{9}\ \bar{8} \\ \bar{7}\ \bar{6}\ \bar{5}\ \bar{4}\ \bar{3} \\ 1\ 2\ \bar{2}\ \bar{1}\ 0 \\ 6\ 4\ 7\ 5\ 3 \\ 12\ 11\ 10\ 9\ 8. \end{cases}$$

In like manner we obtain

$$\begin{cases} \overline{12}\ \overline{11}\ \overline{10}\ \bar{9}\ \bar{8} \\ \bar{2}\ \bar{1}\ 0\ 1\ 2 \\ \bar{4}\ \bar{3}\ \bar{7}\ \bar{6}\ \bar{5} \\ 6\ 4\ 7\ 5\ 3 \\ 12\ 11\ 10\ 9\ 8 \end{cases} \text{ and } \begin{cases} \bar{7}\ \bar{6}\ \bar{5}\ \bar{4}\ \bar{3} \\ 8\ 9\ 10\ 11\ 12 \\ 1\ 2\ \bar{2}\ \bar{1}\ 0 \\ \bar{9}\ \overline{11}\ \bar{8}\ \overline{10}\ \overline{12} \\ 7\ 6\ 5\ 4\ 3, \end{cases}$$

and so forth.

Reverting to the general arrangement of σ zero columns of ρ terms each, and putting $N=\rho\sigma$, that is $2n+1=\overline{2r+1}.\overline{2s+1}$, and therefore $n=2rs+r+s$, we ask attention to the fact that there is in the system one row consisting of all the terms that are greater than $n-\sigma$ or $2rs+r-s-1$; two rows consisting of all the terms greater than $n-2\sigma$ or $2rs+r-3s-2$; three rows of all that are greater than $n-3\sigma$; and so forth.

e.g. In the example above, there is one row consisting of all the terms greater than $12-5$ or 7 and two consisting of those which are greater than $12-10$ or 2.

It follows, that every column contains only one number greater than $n-\sigma$, two greater than $n-2\sigma$, three greater than $n-3\sigma$, and so on. Suppose any one of these zero columns arranged in descending order to be $\nu_1, \nu_2, \nu_3 \ldots \nu_\rho$, and that we increase each of these terms by σ and then diminish $\nu_1+\sigma$ by N or $\rho\sigma$. The results still make a zero series. The same is true if we increase each ν by 2σ and then

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diminish the two terms $\nu_1 + 2\sigma$ and $\nu_2 + 2\sigma$ each by N, or again if we increase each ν by 3σ, and then diminish the three terms $\nu_1 + 3\sigma$, $\nu_2 + 3\sigma$, $\nu_3 + 3\sigma$ each by N; and so forth. Now what has been said amounts to proving that the several series

$$R_N\nu_1 + \sigma, \quad R_N\nu_2 + \sigma, \quad \ldots R_N\nu_\rho + \sigma,$$
$$R_N\nu_1 + 2\sigma, \quad R_N\nu_2 + 2\sigma, \ldots R_N\nu_\rho + 2\sigma,$$
$$R_N\nu_1 + 3\sigma, \quad R_N\nu_2 + 3\sigma, \ldots R_N\nu_\rho + 3\sigma,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

are all zero series. For instance, take from the numerical example above any one of the zero series, as $\bar{5}$, 10, $\bar{2}$, $\bar{8}$, 5. Here $R_{25}\bar{5} + 5$, $R_{25}10 + 5$, and so forth, gives the zero series 0, 10, 3, 3, 10. $R_{25}\bar{5} + 10$, $R_{25}10 + 10$, &c. gives 5, $\bar{5}$, 8, 2, $\bar{10}$, and so forth. This remarkable theorem will find its application in the sequel. Of course the pairs r and s, ρ and σ may both together change places in the investigation.

We are now prepared to discuss the case reserved in No. I, viz. the zero squares dependent on the form $R_N ax + by$, when either a or b or each of them apart is commensurable with N.

Put $N = \rho\sigma$, and for b write $b\sigma$, and suppose ρ and b prime to each other. Then the formula becomes $R_N ax + b\sigma y$. Generally, for $R_N a(\xi + l) + b\sigma(\eta + m)$ write $\mu_{l+1, m+1}$; then putting $x = \xi, \xi+1, \ldots \xi+N-1$, and $y = \eta, \eta+1, \ldots \eta+N-1$ sucessively, we form a square which consists of σ repetitions of the ρ rows following:

$$\begin{array}{c|c|c}
\mu_{1,1}\ \mu_{2,1}\cdots\mu_{\sigma,1} & \mu_{\sigma+1,1}\cdots\mu_{2\sigma,1} & \mu_{2\sigma+1,1}\cdots\mu_{N,1} \\
\mu_{1,2}\ \mu_{2,2}\ \ \mu_{\sigma,2} & \mu_{\sigma+1,2}\ \ \mu_{2\sigma,2} & \mu_{2\sigma+1,2}\ \ \mu_{N,2} \\
\vdots & & \\
\mu_{1,\rho}\ \mu_{2,\rho}\ \ \mu_{\sigma,\rho} & \mu_{\sigma+1,\rho}\ \ \mu_{2\sigma,\rho} & \mu_{2\sigma+1,\rho}\ \ \mu_{N,\rho}
\end{array}$$

Now as b is prime to ρ, we know that some one term, not the first, of the series $\frac{a}{\rho}, \frac{a+b}{\rho}, \ldots \frac{a+\rho-1\,b}{\rho}$ has a zero remainder. Suppose this to be the $(k+1)^{\text{th}}$, viz. $\frac{a+kb}{\rho}$. Then

$$\begin{aligned}
&R_N a(\xi + p\sigma + l) + b\sigma(\eta + pk + m) \\
&= R_N a(\xi + l) + b\sigma(\eta + m) + p\sigma(a + bk) \\
&= R_N a(\xi + l) + b\sigma(\eta + m).
\end{aligned}$$

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That is $$\mu_{\rho\sigma+l+1,\,\rho k+m+1} = \mu_{l+1,\,m+1}.$$

This relation shews that in the $(k+1)^{\text{th}}$, $(k+2)^{\text{th}}$, ... rows of the square beginning at the $\sigma+1^{\text{th}}$ column, we meet with the series

$$\mu_{1,1},\ \mu_{2,1}\cdots\mu_{\sigma,1},$$
$$\mu_{1,2},\ \mu_{2,2}\cdots\mu_{\sigma,2},$$
$$\vdots$$

and again in the $(2k+1)^{\text{th}}$, $(2k+2)^{\text{th}}$ ... rows, beginning at the $2\sigma+1^{\text{th}}$ column, and so forth. We find therefore that the ρ rows above written are separable into ρ parallelograms of σ elements in rank and ρ in file, all consisting of the same rows but circulating, so that the first row of the first parallelogram is the $k+1^{\text{th}}$ of the second, the $2k+1^{\text{th}}$ of the third, and so on. We of course reject multiples of ρ out of any such numero as $pk+m+1$, in the circulation.

We next proceed to shew that each of these parallelograms consists of a certain arrangement of the series

$$-n,\ -n+1,\ \ldots n-1,\ n.$$

For, as the numbers in each parallelogram are the same, the ρ parallelograms consist of the numbers in any one parallelogram ρ times repeated. But as these parallelograms make up ρ rows, each of which is the series

$$-n,\ -n+1,\ \ldots n-1,\ n,$$

in a certain order, they consist altogether of this series ρ times repeated. But it has been said that they consist of the numbers in each parallelogram ρ times repeated. Consequently the numbers in each parallelogram are the series

$$-n,\ -n+1,\ \ldots n-1,\ n. \quad \text{Q.E.D.}$$

Ex. $R_{15}ax+3by$, where $a=2$, $b=4$, $x=1, 2, 3, \ldots$, $y=1, 2, 3, \ldots$, gives the square formed by repeating 3 times the 5 parallelograms below:

$\bar{1}\ 1\ 3\ \ 5\ 7\ \bar{6}\ \ \bar{4}\ \bar{2}\ 0\ \ 2\ 4\ 6\ \ \bar{7}\ \bar{5}\ \bar{3}$

$\bar{4}\ \bar{2}\ 0\ \ 2\ 4\ 6\ \ \bar{7}\ \bar{5}\ \bar{3}\ \ \bar{1}\ 1\ 3\ \ 5\ 7\ \bar{6}$

$\bar{7}\ \bar{5}\ \bar{3}\ \ \bar{1}\ 1\ 3\ \ 5\ 7\ \bar{6}\ \ \bar{4}\ \bar{2}\ 0\ \ 2\ 4\ 6$

$5\ 7\ \bar{6}\ \ \bar{4}\ \bar{2}\ 0\ \ 2\ 4\ 6\ \ \bar{7}\ \bar{5}\ \bar{3}\ \ \bar{1}\ 1\ 3$

$2\ 4\ 6\ \ \bar{7}\ \bar{5}\ \bar{3}\ \ \bar{1}\ 1\ 3\ \ 5\ 7\ \bar{6}\ \ \bar{4}\ \bar{2}\ 0.$

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The general conditions determined in No. I for the grave and acute diagonals were

$$R_N(a+b)x+b(\eta-\xi) \text{ and } R_N(a-b)x+b(\eta+\xi-1),$$

which become in the present case

$$R_N(a+\sigma b)x+\sigma b(\eta-\xi) \text{ and } R_N(a-\sigma b)x+\sigma b(\eta-\xi-1).$$

where $N=\rho\sigma$. Here we can never have $R_N a+\sigma b=0$ nor $R_N a-\sigma b=0$, because a is supposed prime to N; but we may possibly have $a+\sigma b$ some multiple of ρ_1, where ρ_1 is either ρ or some measure of it. In this event the grave diagonal will consist of ρ_1 identical periods. A like remark applies to $a-\sigma b$ and the acute diagonal. Thus in the last example $R_{15}2x+12y$, $a-\sigma b=-12$ contains the factor 3 in common with 15, and the acute diagonal is 2, 7, $\bar{3}$, five times repeated.

Returning now to consider the square above constructed, its first parallelogram is

$$\begin{matrix} \mu_{1,1}, & \mu_{2,1}\cdots\mu_{\sigma,1}, \\ \mu_{1,2}, & \mu_{2,2} \quad \mu_{\sigma,2}, \\ \vdots & \\ \mu_{1,\rho}, & \mu_{2,\rho} \quad \mu_{\sigma,\rho}, \end{matrix}$$

being the series $-n$, $-n+1$, $\ldots n-1$, n in a certain order. First, suppose that neither $a+\sigma b$ nor $a-\sigma b$ is commensurable with N. Let the same series arranged as any parallelogram of σ zero columns of ρ terms each, according to the method already given, be

$$\begin{matrix} \nu_{1,1}, & \nu_{2,1}\cdots\nu_{\sigma,1}, \\ \nu_{1,2}, & \nu_{2,2} \quad \nu_{\sigma,2}, \\ \vdots & \\ \nu_{1,\rho}, & \nu_{2,\rho} \quad \nu_{\sigma,\rho}. \end{matrix}$$

Then if throughout the square we exchange each μ for its corresponding ν, we form a zero square. For every row as well as each diagonal of the square still consists of the series $-n$, $-n+1$, $\ldots n-1$, n, only in a new order. Also every column of ρ terms is the same as one of the first σ zero columns in a different order. And further, by combining the terms of this square by juxtaposition with those of

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another zero square $R_N a_1 x + b_1 y$ according to No. I, we form a zero magic square. For, we reproduce exactly the N^2 permutations of N things taken in pairs with repetitions. Thus, supposing k to be any one element of the latter square, we have in either of the two squares which combines $R_N ax + \sigma by$ and $R_N a_1 x + b_1 y$, the pairs $k, -n$ and $-n, k$; $k, -n+1$ and $-n+1, k$; ... k, n and n, k; and after the elements $-n, -n+1, \ldots n-1, n$ have all been permuted in the process of forming a zero square from $R_N ax + \sigma by$, all these permutations still exist, as before, only in other situations.

We cannot of course combine two squares derived from $R_N ax + \sigma by$ and $R_N a_1 x + \sigma b_1 y$ because N and the coefficients of y have a common measure.

Again, by the theorem above it will be seen that if in the zero square formed from $R_N ax + b\sigma y$, we substitute $R_N \nu + k\sigma$ for every element ν, where k is any integer, we still have a zero square. That is, from any zero square for $R_N ax + b\sigma y$ may be at once derived another for $R_N ax + \sigma(by + k)$, because each of the rows and diagonals still consists of the series $-n, -n+1, \ldots n-1, n$, in new situations; also by the theorem the zero columns still remain such.

But next suppose $a + \sigma b$ to contain a factor ρ, common to N; then we cannot choose any zero parallelogram of ν's at pleasure, but only one so arranged that the period of $\frac{N}{\rho}$ terms $\nu_{11}, \nu_{22}, \nu_{33}, \ldots$ in the first parallelogram (prolonged by repetition if necessary) shall be a zero period. And further, as the theorem does not apply to *such* zero periods, we cannot derive the zero square for $R_N ax + \sigma(by + k)$ from that for $R_N ax + b\sigma y$, as in the former case. The same remarks apply to $a - \sigma b$ and the diagonal $\nu_{\rho,1}, \nu_{\rho-1,2}, \nu_{\rho-2,3}, \ldots$.

Examples.

(1) $R_{15} 2x + 9y$, when $\xi = 1$, $\eta = 1$, gives

$\bar{4}$	$\bar{2}$	0	2	4	6	$\bar{7}$	$\bar{5}$	$\bar{3}$	$\bar{1}$	1	3	5	7	6
5	7	$\bar{6}$	$\bar{4}$	$\bar{2}$	0	2	4	6	$\bar{7}$	$\bar{5}$	$\bar{3}$	$\bar{1}$	1	3
$\bar{1}$	1	3	⋮			⋮			⋮			⋮		
$\bar{7}$	$\bar{5}$	$\bar{3}$												
2	4	6												
⋮														

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Take any parallelogram of 3 zero columns of 5 terms each formed as above, as

$$\begin{matrix} 5 & 6 & 7 \\ 2 & 3 & 4 \\ 0 & 1 & \bar{1} \\ \bar{2} & \bar{4} & \bar{3} \\ \bar{5} & \bar{6} & \bar{7} \end{matrix}$$

Then the series of permutations will be

$$\begin{pmatrix} \bar{4} & \bar{2} & 0 & 5 & 7 & \bar{6} & \bar{1} & 1 & 3 & \bar{7} & \bar{5} & \bar{3} & 2 & 4 & 6 \\ 5 & 6 & 7 & 2 & 3 & 4 & 0 & 1 & \bar{1} & \bar{2} & \bar{4} & \bar{3} & \bar{5} & \bar{6} & \bar{7} \end{pmatrix},$$

giving the zero square

$5\ 6\ 7$	$\bar{5}\ \bar{6}\ \bar{7}$	$\bar{2}\ \bar{4}\ \bar{3}$	$0\ 1\ \bar{1}$	$2\ 3\ 4$
$2\ 3\ 4$	$5\ 6\ 7$	$\bar{5}\ \bar{6}\ \bar{7}$	$\bar{2}\ \bar{4}\ \bar{3}$	$0\ 1\ \bar{1}$
$0\ 1\ \bar{1}$	$2\ 3\ 4$	$5\ 6\ 7$	$\bar{5}\ \bar{6}\ \bar{7}$	$\bar{2}\ \bar{4}\ \bar{3}$
$\bar{2}\ \bar{4}\ \bar{3}$	$0\ 1\ \bar{1}$	$2\ 3\ 4$	$5\ 6\ 7$	$\bar{5}\ \bar{6}\ \bar{7}$
$\bar{5}\ \bar{6}\ \bar{7}$	$\bar{2}\ \bar{4}\ \bar{3}$	$0\ 1\ \bar{1}$	$2\ 3\ 4$	$5\ 6\ 7$
⋮				

Hence we derive the zero square for $R_{15}2x+3(3y+1)$ as underneath:

$\bar{7}\ \bar{6}\ \bar{5}$	$\bar{2}\ \bar{3}\ \bar{4}$	$1\ \bar{1}\ 0$	$3\ 4\ 2$	$5\ 6\ 7$
$5\ 6\ 7$	$\bar{6}\ \bar{7}\ \bar{5}$	$\bar{2}\ \bar{3}\ \bar{4}$	$1\ \bar{1}\ 0$	$3\ 4\ 2$
$3\ 4\ 2$	⋮	⋮	⋮	⋮
$1\ \bar{1}\ 0$				
$\bar{2}\ \bar{3}\ \bar{4}$				
⋮				

(2) $R_{15}x+by$, $\xi=1$, $\eta=1$, gives

$7\ \bar{7}\ \bar{6}$	$\bar{5}\ \bar{4}\ \bar{3}$	$\bar{2}\ \bar{1}\ 0$	$1\ 2\ 3$	$4\ 5\ 6$
$\bar{2}\ \bar{1}\ 0$	⋮	⋮	⋮	⋮
$4\ 5\ 6$				
$\bar{5}\ \bar{4}\ \bar{3}$				
$1\ 2\ 3$				
⋮				

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Here the acute diagonal has the period $1\ \bar{4}\ 6$, and therefore in arranging the zero columns, we must provide that this be replaced by a zero period. We take the same arrangement as before, the order of the numbers in the columns being slightly altered, viz.

$$\begin{array}{c} 5\ 6\ 7 \\ 2\ 3\ \bar{1} \\ 0\ \bar{4}\ 4 \\ \bar{2}\ 1\ \bar{3} \\ \bar{5}\ \bar{6}\ \bar{7} \end{array}$$

Thus the scheme of permutations is

$$\begin{pmatrix} 7\ \bar{7}\ \bar{6} & \bar{2}\ \bar{1}\ 0 & 4\ 5\ 6 & \bar{5}\ \bar{4}\ \bar{3} & 1\ 2\ 3 \\ 5\ 6\ 7 & 2\ 3\ \bar{1} & 0\ \bar{4}\ 4 & \bar{2}\ 1\ \bar{3} & \bar{5}\ \bar{6}\ \bar{7} \end{pmatrix},$$

and we obtain the square

$$\begin{array}{ccccc} 5\ 6\ 7 & \bar{2}\ 1\ \bar{3} & 2\ 3\ \bar{1} & \bar{5}\ \bar{6}\ \bar{7} & 0\ \bar{4}\ 4 \\ 2\ 3\ \bar{1} & \vdots & \vdots & \vdots & \vdots \\ 0\ \bar{4}\ 4 & & & & \\ \bar{2}\ 1\ \bar{3} & & & & \\ \bar{5}\ \bar{6}\ \bar{7} & & & & \\ \vdots & & & & \end{array}$$

But we cannot derive from this, as in the preceding example, the zero square for $R_{15}x + 3(2y + k)$. Thus, if $k = 1$, we should have in the acute diagonal the non-zero period $\bar{4}, 2, 5$.

It is manifest that the method just employed may be applied to the excepted case in No. I., where $a + b$ or $a - b$ or each of them, is commensurable with N; as in $R_{15}2x + 7y$; but a numerical example would occupy too much valuable space.

For the form $R_N\sigma ax + by$, it is obvious that we have to proceed exactly as for $R_N ax + \sigma by$, only putting rows for columns and columns for rows throughout the process.

Suppose next that we have the form $R_N\sigma ax + \tau by$, where $N = \rho\sigma\tau$, σ and τ being according to the original assumption prime to each other.

The preceding investigation shews that the square resolves itself into $\rho\sigma$ rows recurring τ times, and that each set of

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these rows breaks up in $\rho\sigma$ parallelograms of elements τ in rank, $\rho\sigma$ in file, thus separating the whole square into $\rho\sigma\tau$ or N parallelograms τ in rank, $\rho\sigma$ in file, each parallelogram exactly comprising the N elements $-n, -n+1, \ldots n-1, n$. The same investigation shews in like manner that the whole square separates into N parallelograms $\rho\tau$ in rank, σ in file, each parallelogram comprising the elements $-n, -n+1, \ldots n-1, n$. From both these statements together, we infer that the square separates into ρN smaller parallelograms, τ in rank, σ in file; and that ρ of these parallelograms taken either in row or in column, exactly contain the elements $-n, -n+1, \ldots n-1, n$. We may therefore proceed to the corresponding zero square by forming the series $-n, -n+1, \ldots n-1, n$ into either $\rho\tau$ zero columns of σ terms each, or $\rho\sigma$ zero rows of τ terms each. The reader will easily illustrate what has been said upon this case by such an example as

$$R_{105}3ax + 5by, \quad R_{105}3ax + 7by, \quad \text{or} \quad R_{105}5ax + 7by.$$

The last two would have partial periods in the grave diagonal. Though the simplest of their kind, they are too long for insertion here, even in the most abridged form.

Everton Vicarage, near St. Neots.

NOTE ON A SYSTEM OF ALGEBRAICAL EQUATIONS.

By Prof. CAYLEY.

CONSIDER the system of equations

$$a + b(y+z)^2 + cy^2z^2 = 0,$$
$$a + b(z+x)^2 + cz^2x^2 = 0,$$
$$a + b(x+y)^2 + cx^2y^2 = 0,$$

which is a particular case of that belonging to the porism of in-and-circumscribed triangle. We have y and z the roots of

$$a + bx^2 + 2u.bx + u^2(b + cx^2) = 0.$$

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Consequently

$$y+z=\frac{-2bx}{b+cx^2},$$

$$yz=\frac{a+bx^2}{b+cx^2},$$

or substituting in the equation between y and z, this becomes

$$(ac+b^2)(a+4bx^2+cx^4)=0,$$

so that if $ac+b^2$ is not $=0$, we have

$$a+4bx^2+cx^4=0,$$

and moreover

$$(x-y)(x-z)$$

$$=x^2+\frac{2bx^2}{b+cx^2}+\frac{a+bx^2}{b+cx^2}$$

$$=\frac{1}{b+cx^2}(a+4bx^2+cx^4)=0,$$

so that $x=y$ or else $x=z$. If $x=z$, the three equations reduce themselves to the two

$$a+bx^2+2y.bx+y^2(b+cx^2)=0,$$

$$a+4bx^2+cx^4=0,$$

giving $y=x$, or else $y=-\frac{3bx+cx^3}{b+cx^2}$; and it hence appears that if from this last equation and $a+4bx^2+cx^4=0$ we eliminate x, the result must be $a+4by^2+cy^4=0$. For in the same way that the elimination of y, z from the original three equations gives $a+4bx^2+cx^4=0$, the elimination of x, z from the same three equations will give $a+4by^2+cy^4=0$, so that in any case y is a root of this equation.

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ELECTRO-DYNAMICS.

By Percival Frost, M.A.

Ampere, in his theory of Electro-Magnetics, claims the credit of having thrown over the hypotheses of philosophers, which contradicted Newton's law that Action and Reaction were opposite as well as equal; and his formula for the action of elements of currents upon one another, combined with his hypothesis that every magnet acts as a solenoid, meets all the cases of mutual action of voltaic currents and magnets which are at rest. But the main point, upon which he rests his assertion, although very distinctly stated by him, is passed over cursorily by most writers who reproduce his arguments, and seems to be scarcely appreciated by others. I have endeavoured in this paper to make in a simple manner the calculations which are sufficient to establish the mode of action of currents upon currents; and to shew the identity of the action of magnetic fluids, distributed according to Ampere's supposition, with the actions of currents upon the pole of a solenoid or magnet.

I have also pointed out an important error made or quoted by Wiedeman in his work on *Magnetism*, and taken that opportunity of illustrating the different methods of calculating the action of currents of a finite size upon magnets.

Total action of one small plane closed current upon another.

1. Let the plane of the circuit acted on be that of xy, the origin being a point O in the middle of the circuit (fig. 20).

Let the axis of y be parallel to the plane of the acting circuit, the normal to which is inclined at the angle α to Ox.

x, y, z, the coordinates of a point C in the middle of the acting circuit.

x_1, y_1, 0, those of an element ds at P, in the circuit O.

λ, λ' the areas of the circuits C and O

$$l^2 = CO^2 = x^2 + y^2 + z^2$$

$$CP^2 = l^2 - 2(xx_1 + yy_1),$$

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neglecting squares of small quantities;

then $$\int x_1 dy_1 = \lambda', \quad \int y_1 dx_1 = -\lambda_1,$$

ds, being measured in the direction of the current,

also $$\int dx_1 = 0, \quad \int dy_1 = 0, \quad \int x_1 dx_1 = 0, \quad \int y_1 dy_1 = 0.$$

the integrals being taken over the whole circuit.

The formula for the action of a current on an element resolved parallel to the axis of z, p being the perpendicular from O on the plane of λ, is

$$\tfrac{1}{2} ii' \lambda \frac{d}{dp} \left\{ \frac{(x-x_1)\,dy_1 - (y-y_1)\,dx_1}{r^3} \right\}.$$

If dZ be the action on ds_1, parallel to Oz, neglecting squares of small quantities

$$dZ = \tfrac{1}{2} ii' \lambda \frac{d}{dp} \left\{ \frac{(x-x_1)\,dy_1 - (y-y_1)\,dx_1}{l^3} \left(1 + 3\frac{xx_1 + yy_1}{l^2}\right) \right\}$$

$$= \tfrac{1}{2} ii' \lambda \frac{d}{dp} \left\{ \frac{(x-x_1)\,dy_1 - (y-y_1)\,dx_1}{l^3} + 3\frac{xx_1 + yy_1}{l^5} (x dy_1 - y dx_1) \right\},$$

therefore, for the action on the complete circuit λ'

$$Z = \tfrac{1}{2} ii' \lambda \frac{d}{dp} \left\{ \frac{-2\lambda'}{l^3} + 3\frac{x^2\lambda' + y^2\lambda'}{l^5} \right\}$$

$$= \tfrac{1}{2} ii' \lambda\lambda' \frac{d}{dp} \left(\frac{1}{l^3} - \frac{3z^2}{l^5} \right)$$

$$= \tfrac{1}{2} ii' \lambda\lambda' \frac{d^2}{dp\,dz} \left(\frac{z}{l^3} \right),$$

since $$\frac{dl}{dz} = \frac{z}{l},$$

$$= -\tfrac{1}{2} ii' \lambda\lambda' \frac{d^2}{dp\,dx} \left(\frac{dV}{dz} \right),$$

when V is the potential of the area of the circuit λ with respect to the point O, for if C be the centre of gravity, and squares of small quantities be neglected, and ξ, η, ζ be the coordinates of dA with reference to axes through C

$$V = \int \frac{dA}{l} \left(1 - \frac{x\zeta + y\eta + z\xi}{l^2} \right) = \frac{\lambda}{l}.$$

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Again $$dX = -\tfrac{1}{2}ii''\lambda \frac{d}{dp}\frac{z dy_1}{l^3}\left(1 + 3\frac{xx_1 + yy_1}{l^2}\right)$$

$$X = -\tfrac{1}{2}ii''\lambda\lambda' \frac{d}{dp}\frac{3xz}{l^5}$$

$$= \tfrac{1}{2}ii''\lambda\lambda' \frac{d^2}{dp\,dx}\left(\frac{z}{l^3}\right)$$

$$= -\tfrac{1}{2}ii''\lambda\lambda' \frac{d^3}{dp\,dx}\left(\frac{dV}{dz}\right).$$

Similarly $$Y = -\tfrac{1}{2}ii''\lambda\lambda' \frac{d}{dp}\frac{3yz}{l^5}$$

$$= -\tfrac{1}{2}ii'\lambda' y \frac{d}{dp}\left(\frac{dV}{dz}\right).$$

For the components of the couple

$$L = \int y_1 dZ = \int \tfrac{1}{2}ii''\lambda y_1 \frac{d}{dp}\left(\frac{x dy_1 - y dx_1}{l^3}\right)$$

$$= \tfrac{1}{2}ii''\lambda\lambda' \frac{d}{dp}\left(\frac{y}{l^3}\right)$$

$$= -\tfrac{1}{2}ii''\lambda' \frac{d}{dp}\left(\frac{dV}{dy}\right)$$

$$M = -\int x_1 dZ = -\tfrac{1}{2}ii''\lambda\lambda' \frac{d}{dp}\frac{x}{l^3}$$

$$= \tfrac{1}{2}ii''\lambda' \frac{d}{dp}\left(\frac{dV}{dx}\right)$$

$$N = \int (x_1 dY - y_1 dX) = 0.$$

2. It can easily be shown by way of testing these results, that the reversed action of these forces gives the action of the circuit at O on that at C, in a similar form.

This is obvious for the single forces, by the results quoted in the last number of the *Journal*, since the components depend upon an expression of the form

$$\frac{d^2}{dp\,dp'}\left(\frac{x - x'}{l^3}\right),$$

and the reversed couples about the axes are

$$L_1 = -L - Yz + Zy,$$
$$M_1 = -M - Zx + Xz,$$
$$N_1 = -N - Xy + Yx;$$

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therefore if L', M', N' be the components of the reversed couple round the axes of x', y', z', chosen with respect to O in the same manner as those of x, y, z are with respect to C,

$$\begin{aligned} L' = &\ L_1 \cos\alpha - N_1 \sin\alpha \\ = &- L\cos\alpha + N\sin\alpha \\ &- Yz' + y'(Z\cos\alpha + X L\alpha); \end{aligned}$$

therefore

$$L' = \tfrac{1}{2} ii'' \lambda\lambda' \left\{ - \frac{d}{dp}\left(\frac{y}{l^3}\right)\cos\alpha + z' \frac{d}{dp}\left(\frac{3yz}{l^5}\right) + y' \frac{d}{dp}\left(\frac{1}{l^3}\cos\alpha - \frac{3zz'}{l^5}\right),$$

which, since $y = y'$, and $z' = p$, $z = p'$,

$$= - \tfrac{1}{2} ii'' \lambda\lambda' \frac{3y'p'}{l^5}$$

$$= \tfrac{1}{2} ii'' \lambda\lambda' \frac{d}{dp'}\left(\frac{y'}{l^3}\right),$$

and similarly for the others.

Whole action of an infinite solenoid upon a small plane current.

3. Let the same axes be taken as before, let λ be the area of each of the elementary currents of the solenoid, and use the same notation as in Cor. 5 of the last number.

For the single forces

$$X = \tfrac{1}{2} ii'' \frac{\lambda\lambda'}{g} \cdot \frac{3xz}{l^5},$$

$$Y = \tfrac{1}{2} ii'' \frac{\lambda\lambda'}{g} \cdot \frac{3yz}{l^5},$$

$$Z = - \tfrac{1}{2} ii'' \frac{\lambda\lambda'}{g}\left(\frac{1}{l^3} - \frac{3z^2}{l^5}\right).$$

And for the couple

$$L = - \tfrac{1}{2} ii'' \frac{\lambda\lambda'}{g} \frac{y}{l^3},$$

$$M = \tfrac{1}{2} ii'' \frac{\lambda\lambda'}{g} \frac{x}{l^3},$$

$$N = 0.$$

The action of the solenoid upon the current is compounded of a single force $\tfrac{1}{2} ii'' \frac{\lambda\lambda'}{g} \frac{1}{l^2}$ ..... (1),

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in the direction perpendicular to the plane of the circuit, and repulsive;

of a single force $\frac{1}{2}ii'\frac{\lambda\lambda'}{g}\frac{3z}{l^4}$ ..... (2)

tending to the pole of the solenoid;

and of a couple of moment $\frac{1}{2}ii'\frac{\lambda\lambda'}{g}\frac{\sqrt{(x^2+y^2)}}{l^3}$ ..... (3),

whose axis is perpendicular to the plane containing the line joining the pole with the centre of the circuit, and the projection of this line on the plane of the circuit.

The action of the circuit on the solenoid consists of the reversed forces (1) and (2), for the couple vanishes, being compounded of the reversed couple (3), and the couple introduced by the transference of (1) to this pole, the moment of which is the same as that of (3).

4. Gauss' construction for the action of a small bar magnet on a particle of free magnetism, is represented by the expressions for the forces (1) and (2), for, let C (fig. 21) be the pole of the solenoid, O the small circuit, OD normal to it, CD perpendicular to OC, and $OE=\frac{1}{3}OD$.

If OC represent the force (2), $OC\frac{l}{3z}=EO$ will represent force (1), and CE will represent the resultant action of O in direction and magnitude.

On the action and reaction of an element of a current and the pole of a solenoid.

5. From the above investigation of the actions of currents, it follows,

That it is correct to state, that the action of the pole of a solenoid upon an element of a current is a single force acting on the element perpendicular to the plane containing the element and the pole, according to Biot's law.

That it is incorrect to state that the action of the element on the pole is a single force acting at the pole, according to that law, since by Newton's law of Action and Reaction, that action must also be a force acting at the element.

That the correctness of the results obtained in many calculations, as in the case of a straight current, arises from a composition of two errors, the couples which are neglected in the part of the circuit considered, being counteracted by those in other portions of the closed current which are not considered in the calculations.

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For example, if a be the distance of a pole N from a straight current of considerable length, the distances of an element from O, the foot of the perpendiculars from N, being x; the action of the element, taken to be at N, and perpendicular the plane containing the current and pole, and of a magnitude $\frac{mi dx}{2r^2} \cdot \frac{a}{r}$, gives the total action of the current

$$\int_{-\infty}^{\infty} \frac{mia\,dx}{2(a^2+x^2)^{\frac{3}{2}}} = \tfrac{1}{2}mia \left\{ \frac{1}{a^2} \frac{x}{\sqrt{(a^2+x^2)}} \right\}_{-\infty}^{+\infty} = \frac{mi}{a}.$$

The resultant of all such couples as $\frac{mia}{2r^2}\,dx$, equivalent to $\frac{mixa}{2r^3}\,dx$ and $\frac{mia^2}{2r^3}\,dx$ is a couple mi, whose axis is parallel to the current, and this ought to be introduced as part of the action of the current; but its omission does not affect the result, because in the return current supposed at a great distance and parallel to the part considered, another couple $-mi$ would have been omitted, as well as the single force which would be properly neglected, because its magnitude varies as the reciprocal of the distance.

6. The method of treating such a current as a closed current, by spreading small plane circuits over the surface extended to a great distance, gives the proper result in a proper way, for each small circuit has been shown to act as a single force at N, perpendicular to the plane. The result obtained this way is, taking Oy in the line NO produced,

$$-\int_0^{\infty}\int_{-\infty}^{+\infty} \frac{mi\,dx\,dy}{2\{(x+a)^2+y^2\}^{\frac{3}{2}}}$$

$$= -\int_0^{\infty} \frac{mi}{2(x+a)^2} \left\{ \frac{y}{\{(x+a)^2+y^2\}^{\frac{1}{2}}} \right\}_{-\infty}^{\infty} = -\int_0^{\infty} \frac{mi}{(x+a)^2} = \frac{mi}{a}.$$

To shew that any small plane current may be replaced by a small magnet placed with its centre in the centre of gravity of the current, and perpendicular to the plane of the current.

7. Let the element be at O (fig. 22), and the direction of the current Ox, x, y, z the coordinates of a point in which a mass μ of north magnetism resides, PM perpendicular to the plane xy, MN to OX, LP to the plane NOP.

LP is the direction of the action of the force

$$\frac{\mu}{OP^2} \cdot \frac{PN}{OP}\, i'' dx_1 \ldots\ldots\ldots\ldots\ldots (1),$$

by Biots's law.

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The component in MP is $\frac{\mu y}{r^3} i' dx_1$.

If a magnet of length 2ε be placed with its centre at P in any direction, such that p is the perpendicular from O on the plane normal to the magnet through P, the action of the two poles of the magnet revolved parallel to Oz, are

$$\mu i' dx_1 \left\{ \frac{y}{r^3} + \varepsilon \frac{d}{dp} \left(\frac{y}{r^3} \right) \right\},$$

and

$$-\mu i' dx_1 \left\{ \frac{y}{r^3} - \varepsilon \frac{d}{dp} \left(\frac{y}{r^3} \right) \right\},$$

and the component parallel to $Oz = 2\mu\varepsilon i'\, dx_1 \frac{d}{dp} \left(\frac{y}{r^3} \right)$, which is the same as that of a closed circuit of area λ, if $\mu . 2\varepsilon$ the magnetic moment $= \frac{1}{2} i\lambda$.

8. Cor. 1. The action of a solenoid, whose axis is any curve, upon an element of a current is the same as that of an infinite number of small magnets ranged along the curve, the magnetic moment of each of which is $\frac{1}{2} i\lambda$. When the solenoid is infinite, x, y, z the coordinates of the pole, the action upon the element dx_1 is $2\mu\varepsilon i' dx_1 \frac{y}{gr^3}, \frac{dp}{g}$ being the number of small magnets in the distance dp, which, by reference to (1), is the same as the action of the north pole of a magnet, if $\frac{2\mu\epsilon}{g}$ or $\frac{i\lambda}{2g}$ represent the quantity of north magnetism, supposed collected in the pole.

9. Cor. 2. The action of the small magnet on the element being the same as that of a small plane circuit, it follows that the single force and couple obtained by the action on a plane current made up of such elements, will be the same for the small magnet as for the small current.

Methods of calculating the action of a given closed current.

10. The actions of a given closed current of any form may be calculated either directly or by the artifice of supposing it the boundary of any arbitrary surface, which surface is divided into small elements, the boundary of each of the elements being ultimately a small closed plane circuit of the same intensity as that of the given circuit and in the same direction.

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Each elementary circuit which is within the boundary, being surrounded by elements in portions of which the streams are moving in contrary directions, has its action counteracted by these adjacent portions, and it is only those which touch the arc of the given circuit which have the portions in contact uncounteracted, and the total action of such portions is the same as that of the given circuit.

Or, instead of each of these elementary circuits we can imagine a small magnet whose magnetic moment is $\frac{1}{2}i\lambda$ placed perpendicular to the element, and then the resultant action of these magnets may be calculated directly, or we may suppose the magnetism of each element distributed upon and below the arbitrary surface, so as to form two layers of magnetism of uniform intensity in each layer, forming surfaces parallel to the surface which fills the space within the given circuit, and then calculate the effect on the pole of a solenoid or magnet, by means of the potentials of the two layers with respect to the point.

11. We can shew by direct calculation of the action as follows that the surface which is interposed is arbitrary, with certain limitations.

For, if we take two such arbitrary surfaces A and B, (fig. 23) the currents distributed over A in the direction of the main current produce the same effect as currents distributed over A and at the same time two sets of currents on B in opposite directions, and these three sets are equivalent to a set in the same direction as the main current on B, and a set all in one direction in the closed surface formed by A and B together, and it only remains to shew that this last has no effect on the pole of the magnet.

Let f, g, h, be the coordinates of the pole m acted on by the currents, λ, λ' small areas intercepted on the closed surface by a cone of small solid angle whose vertex is m, dn the small normal measured outwards at λ, $m\lambda = r$, $m\lambda' = r'$, the action on m of the current round λ, resolved parallel to Oz, is

$$C\lambda \frac{d}{dn}\left(\frac{z-h}{r^3}\right) = C\lambda \frac{d}{dh}\frac{d}{dn}\frac{1}{r}$$

$$= C\frac{d}{dh}\left(-\frac{\lambda}{r^2}\frac{dr}{dn}\right)$$

$$= C\frac{d}{dh}\frac{\lambda\cos\phi}{r^2},$$

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where ϕ is the acute angle between $m\lambda$ and the normal;

for the action of λ', $\frac{dr'}{dn'} = \cos\phi'$,

and the current is in the opposite direction with reference to m,

also, $$\frac{\lambda \cos\phi}{r^2} = \frac{\lambda' \cos\phi'}{r'^2};$$

therefore the whole action parallel to Oz of the two currents round λ, λ' is nil. Hence, the action of all the currents distributed over the closed surface is nil, unless m be within the surface.

12. I shall illustrate these four methods by taking an example which will point out at the same time the error to which I have alluded.

The error is as follows:

The action of AB (fig. 24) a closed circuit upon the pole of a magnet is required, and the method employed is to make a conical surface of which the pole m is the vertex and the closed circuit the guiding curve, to describe a sphere with centre m, CD being the portion intercepted by the curve; the surface which is then selected as the one whose boundary is to be the closed curve AB, is that made up of the portion $ACDB$ of the sheet of the cone and of the spherical segment CD, this surface is divided into elements λ, and small magnets whose moments are $\frac{1}{2}i\lambda$, are placed perpendicular to each element.

It is asserted that the magnets on the sheet $ACDB$ produce no action on m, and then the only acting magnets are those on CD.

This is to me obviously not the case; for if $ns = 2\varepsilon$ (fig. 25) be the magnet perpendicular to λ in the plane of which m lies the repulsion of n on m, supposed a north pole, is $\frac{i\lambda}{4\varepsilon} M\frac{1}{nm^2}$ the attraction of s is $\frac{i\lambda}{4\varepsilon} M\frac{1}{sm^2}$, and the resultant action is perpendicular to $m\lambda$, and equal to $\frac{i\lambda}{2\varepsilon}\frac{1}{m\lambda^2}\frac{\varepsilon}{m\lambda}$; or, if we consider directly the action of the plane circuit λ, it is $\frac{1}{2}i\lambda\frac{1}{m\lambda^2}$, so that the action of the circuits or small magnets which correspond to $ACDB$ is in no case nil.

I shall now take a particular case and illustrate the different methods.

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To calculate the mutual action of a circular current and a pole of a magnet.

13. I. To calculate directly by Biot's law.

Let $mC=a$, $mA=r$, (fig. 26) and let the closed circuit be a circle, 2α the angle of the cone. The action of m on the element $r\sin\alpha\, d\phi$ is $\frac{1}{2}mi\frac{r\sin\alpha\, d\phi}{r^2}\sin\alpha$ in the direction of the axis. The total action is $\pi mi\frac{\sin^2\alpha}{r}$.

II. Calculation by means of currents distributed over a spherical segment whose base is AB, centre m.

Since, by Art (3), the whole action of a current λ on the pole in the normal to the plane of the current, being the difference of the forces (1) and (2) is $\frac{ii'\lambda\lambda'}{gl^2}=\frac{im\lambda}{l^3}$; the component in the direction of the axis of the action of m on an element dS whose normal is inclined at an angle θ to the axis is $\frac{im}{r^3}dS.\cos\theta$; therefore, since $dS\cos\theta$ is the orthogonal projection of dS on the plane of AB, the total action in the axis is $\frac{\pi imr^2\sin^2\alpha}{r^3}=\frac{\pi im\sin^2\alpha}{r}$.

III. To calculate by currents on the surface $ACDB$.

The action of a current λ on a pole in its plane being $\frac{im\lambda}{2l^2}$, the action of the currents on the conical portion resolved in the direction of the axis, having regard to the direction of the small currents, is

$$= \int_a^r\int_0^{2\pi}\frac{imr\sin\alpha\, d\theta\, dr}{2r^3}\sin\alpha$$

$$= -\int_a^r\frac{\pi im\sin^2\alpha\, dr}{r^2}$$

$$= \pi im\sin^2\alpha\left(\frac{1}{r}-\frac{1}{a}\right),$$

that of the spherical portion $=\pi im\sin^2\alpha\frac{1}{a}$, the action of the given current is therefore $\frac{\pi im\sin^2\alpha}{r}$.

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IV. To calculate by means of the potential of the magnetic fluid distributed over two spherical surfaces near AB at distances ε on each side.

If ρ, ρ' be the quantities of attracting and repelling magnetic fluids on an unit of each surface.

μ the quantity in one of the poles of the small magnets, corresponding to an area λ, since it is distributed over an area $\lambda\left(\frac{r+\varepsilon}{r}\right)^2$ for the surface radius $r+\varepsilon$,

$$\rho = \frac{\mu}{\lambda}\left(\frac{r}{r+\varepsilon}\right)^2 = \frac{i}{4\epsilon}\left(\frac{r}{r+\varepsilon}\right)^2,$$

and the potential with respect to a point at a distance x from the vertex, is

$$\int \frac{\rho dS'}{r+\varepsilon+x\cos\theta} = \int \frac{i}{4\epsilon}\cdot\frac{dS}{r+\varepsilon+x\cos\theta},$$

dS', dS being corresponding elements of the surfaces whose radii are $r+\varepsilon$, r respectively.

The whole potential V of the two fluids is

$$\frac{i}{4\varepsilon}\int dS\left\{\frac{1}{r-\varepsilon+x\cos\theta} - \frac{1}{r+\varepsilon+x\cos\theta}\right\}$$

$$= \tfrac{1}{2}i\int \frac{dS}{(r+x\cos\theta)^2}$$

$$= \tfrac{1}{2}i\int \frac{dS}{r^2}\left(1 - \frac{2x\cos\theta}{r} + \ldots\right)$$

$$= V_0 - \frac{ix\pi r^2\sin^2\alpha}{r^2} + \ldots,$$

therefore $-\frac{dV}{dx} = \frac{\pi i\sin^2\alpha}{r}$, when $x=0$, which gives the same result as before.

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ON THE EVOLUTE OF CUBIC CURVES.

By Henry M. Jeffery, M.A.

(Continued from p. 81).

5. To find the Boothian equation to the evolute of a non-singular spherical curve of the third class.

Let this curve, as the plane curve in § 4, be denoted by the equation

$$a\xi^3 + b\eta^3 + c + 6d\xi\eta = 0.$$

The equation to its evolute may be deduced from the three-point equation of § 2 by substituting

$$\frac{p}{\xi} = \frac{q}{\eta} = -r = (1+\xi^2+\eta^2)^{-\frac{1}{2}}, \text{ and } \frac{P}{\xi} = \frac{Q}{\eta} = -R = (1+\xi^2+\eta^2)^{-\frac{1}{2}},$$

(*Journal*, Vol. IX., p. 206),

$$(a\xi^3 + b\eta^3 + c + 6d\xi\eta)\{b^4c^2\xi^6 + c^2a^2\eta^6 + a^2b^2 + \ldots\}$$
$$+ (\xi^2+\eta^2+1)^2(8d^3 + abc)\{6d(a\xi\eta^2 + b\xi^2\eta + c\xi^2\eta^2)$$
$$+ (2ac\eta^3 + 2ab - bc\xi^3)\xi^2 + (2ab + 2bc\xi^3 - ac\eta^3)\eta^2$$
$$+ (2bc\xi^3 + 2ac\eta^3 - ab)\} = 0.$$

This is also the Boothian equation to the evolute of a non-singular spherical curve of the third order

$$(-ax^3 - by^3 + c + 6dxy = 0),$$

since the two curves are reciprocal.

6. To determine the Boothian equation to the evolute of a non-singular plane curve of the third order.

Let its Cartesian equation be

$$-ax^3 - by^3 + c + 6dxy = 0.$$

The equation to the evolute may be obtained, from §§ 2, 3, by the following conversion:

$$\frac{p}{\xi} = \frac{q}{\eta} = -r = -(1+\xi^2+\eta^2)^{-\frac{1}{2}}, \quad R = 0, \quad \frac{P}{\xi} = \frac{Q}{\eta},$$

$$(a\xi^3 + b\eta^3)\{b^4c^2\xi^6 + c^2a^2\eta^6 + a^2b^2 - 24bcd^2\xi^4\eta - 24acd^2\xi\eta^4 - 24abd^2\xi\eta$$
$$- 2\eta^3(a^2bc + 16ad^3) - 2\xi^3(ab^2c + 16bd^3) - 2\xi^3\eta^3(abc^2 + 16cd^2)$$
$$- 24\xi^2\eta^2(2d^4 + abcd)\} + (\xi^2+\eta^2)^2(8d^3 + abc)\{6d(a\xi\eta^2 + b\xi^2\eta)$$
$$+ (2ac\eta^3 + 2ab - bc\xi^3)\xi^2 + (2ab + 2bc\xi^3 - ac\eta^3)\eta^2\} = 0.$$

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7. To find the equation to the evolute of a cusped cubic referred to three-point tangential coordinates.

Let its tangential equation be

$$a_1 p^3 + 3b_2 q^2 r = 0,$$

so that the reciprocal cubic is thus denoted:

$$4b_3 a^3 \alpha^3 - 9a_1 b^3 c \beta^2 \gamma = 0.$$

As in § 2 the equation to the evolute is the eliminant of three equations

$$a_1 f^3 + 3b_3 g^2 h = 0,$$

$$a_1 f p^2 + 2b_2 g q r + b_3 h q^2 = 0,$$

$$fP + gQ + hR = 0,$$

where P is written instead of $a(ap - bq\cos C - cr\cos B)$. It is found to be

$$(a_1 p^3 + 3b_2 q^2 r)(4b_3 P^3 - 9a_1 Q^2 R) = a_1 b_3 (4pP - 8qQ + rR)(4\Delta^2)^2.$$

8. To find the equation to the evolute of a cusped cubic referred to trilinear coordinates.

Let its equation be

$$a_1 a^2 \alpha^2 + 3b_3 b^2 c \beta^2 \gamma = 0.$$

The equation to the evolute is seen to be, from the permanence of equivalent forms,

$$(a_1 P^3 + 3b_3 Q^2 R)(4b_3 p^3 - 9a_1 q^2 r) = a_1 b_3 (4pP - 8qQ + rR)(4\Delta^2)^2.$$

If cusped cubics have the same cusp and point of inflexion, and if the tangents at those points intersect in the same point, the common tangents of all these cubics and their evolutes envelop the same parabola. See § 14, Cor.

9. In the Boothian system of coordinates, the evolutes of the cusped cubics $(a_1 \xi^3 + 3b_3 \eta^2 = 0,\ a_1 x^3 + 3b_3 y^2 = 0)$ are thus denoted:

$$(a_1 \xi^3 + 3b_3 \eta^2)\xi^3 - a_1 (\xi^2 + \eta^2)^2 (\xi^2 - 2\eta^2) = 0,$$

$$(4b_3 \xi^3 + 9a_1 \eta^2)\xi^3 - 4b_3 (\xi^2 + \eta)^2 (\xi^2 - 2\eta^2) = 0.$$

In this particular case of the semi-cubical parabola, the equations to the evolute degenerates into one of the fourth degree; the enveloped parabola degenerates into two points, in which the common tangents meet the line at infinity, as

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is shewn in fig. 27; the line at infinity also passes through the points of inflexion of both curve and evolute.*

10. To find the equation to the evolute of a cubic curve of the third class with double tangents.

This curve is also of the fourth order. Let its equation be

$$a_1 p^3 + 3a_2 p^2 r + 3b_2 q^2 r = 0.$$

The equation to its evolute may be written

$$(a_1 p^3 + 3a_2 p^2 r + 3b_2 q^2 r)\,(12a_2 b_2^2 P^4 + 24a_2^2 b_2 P^2 Q^2 + 12a_2^3 Q^4 - 4a_1 b_2^2 P^3 R - 36a_1 a_2 b_2 P Q^2 R + 9a_1^2 b_2 Q^2 R^2)$$
$$= (4\Delta^2)^2 b_2 \{p\,(12a_1 a_2 b_2 P^2 - 4a_1^2 b_2 PR - 12a_1 a_2^2 Q^2) + gQ\,(8a_1^2 b_2 R - 24a_1 a_2 b_2 P) - a_1^2 b_2 r R^2\}.$$

11. To find the equation to the evolute of a cubic curve of the third class with double tangents, referred to Boothian coordinates.

We have to find the eliminant of the three equations (see § 4)

$$a_1 f^3 + 3a_2 f^2 + 3b_2 g^2 = 0,$$
$$\xi\,(a_1 f^2 + 2a_2 f) + 2b_2 g\eta + a_2 f^2 + b_2 g^2 = 0,$$
$$f\xi + g\eta = 0.$$

It is seen to be

$$3\,(b_2\xi^2 + a_2\eta^2)\,(a_1\xi\eta^2 + a_2\eta^2 + b_2\xi^2) = 2a_1\,(a_2 - b_2)\,\xi\eta^4,$$

which may be also thus expressed

$$(a_1\xi^3 + 3a_2\xi^2 + 3b_2\eta^2)\,(a_2\eta^2 + b_2\xi^2)^2 + a_1 b_2\,(\xi^2 + \eta^2)^2\,\{b_2\xi\,(2\eta^2 - \xi^2) + a_2\xi\eta^2\} = 0.$$

12. To determine the evolute of a cubic curve of the third order with double points, referred to Boothian coordinates.

Let the Cartesian equation of the curve be

$$-a_1 x^3 + 3a_2 x^2 + 3b_2 y^2 = 0.$$

* Generally, if the equation to a curve of the n^{th} class be

$$\xi^n + na\eta^{n-1} = 0,$$

its evolute may be denoted by the equation

$$(\xi^n + na\eta^{n-1})\,\xi^n$$
$$= (\xi^2 + \eta^2)^2\,\{\xi^{2n-4} - 2\xi^{2n-6}\eta^2 + \ldots + (-1)^{n-1}\,(n-2)\,\xi^2\eta^{2n-6} + (-1)^n\,(n-1)\,\eta^{2n-4}\}.$$

This equation to the evolute is unaltered in form, if the given equation represent a curve of the n^{th} order.

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We have to eliminate between its equivalent Boothian equation and two others

$$12a_3(a_3g^2+b_3f^2)^2-4a_1b_3^2f^3-36a_1a_3b_3fg^2 + 9a_1^2b_3g^2=0\equiv\phi(f,g).....(1),$$

$$(\xi-f)\frac{d\phi}{df}+(\eta-g)\frac{d\phi}{dg}=0.....(2),$$

$$f\xi+g\eta=0.....(3).$$

Substitution of the value of g from (3) in (1) and (2) leads to two quadratic equations

$$12a_3(a_3\xi^2+b_3\eta^2)^2f^2-4a_1b_3\eta^2(b_3\eta^2+9a_3\xi^2)f+9a_1^2b_3\xi^2\eta^2=0 \equiv Lf^2+Mf+N,$$

$$f^2\{48(b_3-a_3)\xi(a_3\xi^2+b_3\eta^2)-4a_1b_3(b_3\eta^2+9a_3\xi^2)\} +f\{72a_1a_3b_3\xi\eta^2-12a_1b_3\xi(b_3\eta^2+3a_3\xi^2)+18a_1^2b_3\xi^2\} -18a_1^2b_3\xi\eta^2=0\equiv lf^2+mf+n.$$

Their eliminant determines the evolute

$$(Ln-Nl)^2-(Lm-Ml)(Mn-Nm)=0.$$

After rejecting the factor $(3\xi^2a_3-\eta^2b_3)^2$, this may be reduced to the normal form

$$\xi^2\{12a_3(a_3\eta^2+b_3\xi^2)^2-4a_1b_3^2\xi^2-36a_1a_3b_3\xi\eta^2+9a_1^2b_3\eta^2\} +4(\xi^2+\eta^2)^2\{a_1b_3(\xi^2-2\eta^2)+3a_3^2\xi\eta^2+3a_3b_3\xi(2\eta^2-\xi^2)\}=0.$$

13. To determine the evolute of a cubic curve of the third order with double points, referred to trilinear coordinates.

If the equation to the curve be selected as

$$a_1\alpha^3\alpha^3+3a_3a^2c\alpha^2\gamma+3b_3b^2c\beta^2\gamma=0,$$

the tangential equation to its evolute may be deduced from that given in § 10 by interchanging p, q, r with P, Q, R. This conclusion is inferred from the result of the calculation in the preceding paragraph, wherein a superfluous factor is rejected, and from the general consideration of the degree of the equation of an evolute and of its spherical interpretation.

14. To find the equation to the evolute of a cubic of the third class referred to three-point tangential coordinates.

This equation can be expressed in a symmetrical form, after the process adopted in § 2, and afterwards be reduced to the normal form, in which the foci of the cubic and its

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reciprocal curve are shewn to be common to their respective evolutes.

Let the general equation to the cubic be

$$a_1f^3+b_2g^3+c_2h^3+6dfgh+3a_2f^2g+3a_3f^2h+3b_1fg^2+3b_3g^2h+3c_1fh^2+3c_3gh^2=0\equiv\phi(f,g,h)\ldots\ldots\ldots(1).$$

To determine the evolute we have to eliminate between this and two other equations

$$p\frac{d\phi}{df}+q\frac{d\phi}{dg}+r\frac{d\phi}{dh}=0\ \ldots\ldots\ldots\ldots\ (2),$$

$$fP+gQ+hR=0\ \ldots\ldots\ldots\ldots\ldots\ (3).$$

For convenience of the analysis, substitute uncial letters, thus defined:

$$a_1=A_1P^3,\ b_2=P_2Q^3,\ \ldots\ldots=\ldots\ldots.$$

Also use the subsidiary quantities L, M, N, H,

$$L=(A_1-A_2-A_3)Pp+(B_3-B_1+A_2)Qq+(C_2-C_1+A_3)Rr+D(Pp-Qq-Rr),$$

$$M=(A_3-A_2+B_1)Pp+(B_2-B_3-B_1)Qq+(C_1-C_2+B_3)Rr+D(Qq-Rr-Pp),$$

$$N=(A_2-A_3+C_1)Pp+(B_1-B_3+C_2)Qq+(C_3-C_1-C_2)Rr+D(Rr-Pp-Qq),$$

$$-H=MN+NL+LM.$$

As in § 2, the following ratios of fP, gQ, hR may be obtained from (2) and (3),

$$fP\{L-\sqrt{(H)}\}=gQ\{M+\sqrt{(H)}\},$$

$$gQ\{M-\sqrt{(H)}\}=hR\{N+\sqrt{(H)}\},$$

$$hR\{N-\sqrt{(H)}\}=fP\{L+\sqrt{(H)}\}.$$

Hence
$$\frac{fP}{M+\sqrt{(H)}}=\frac{gQ}{L-\sqrt{(H)}}=\frac{hR}{L+M}.$$

By substituting these values, in the given equation (1),

$$-A_1\{M+\sqrt{(H)}\}^3-B_2\{L-\sqrt{(H)}\}^3+C_3(L+M)^3+\ldots=0.$$

By symmetry,

$$A_1(M+N)^3-B_2\{N+\sqrt{(H)}\}^3-C_3\{M-\sqrt{(H)}\}^3+\ldots=0$$

$$-A_1\{N-\sqrt{(H)}\}^3+B_2(N+L)^3-C_3\{L+\sqrt{(H)}\}^3+\ldots=0.$$

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By addition, the equation to the evolute may be written

$$\begin{aligned}
&\sqrt{(H)}\,\{A_1(N^3-M^3)+B_2(L^3-N^3)+C_3(M^3-L^3)\\
&\quad+3(C_2-B_3)(L+N)(L+M)+3(A_3-C_1)(M+L)(M+N)\\
&\quad+3(B_1-A_2)(N+M)(N+L)\}\\
&=A_1(M+N)(H-MN)+B_2(N+L)(H-NL)+C_3(L+M)(H-LM)\\
&\quad-6D(M+N)(N+L)(L+M)+3(B_2N+C_3M)(L+M)(N+L)\\
&\quad+3(C_1L+A_3N)(M+N)(L+M)+3(A_2M+B_1L)(N+L)(N+M).
\end{aligned}$$

After squaring both sides and rejecting the factor

$$(M+N)(N+L)(L+M),$$

this equation takes the following form:

$$\begin{aligned}
&A_1^2(M+N)^3+B_2^2(N+L)^3+C_3^2(L+M)^3-2B_2C_3L(L^2+3H)\\
&-2A_1C_3M(M^2+3H)-2A_1B_2N(N^2+3H)+9A_2^2(M+N)^2(L+N)\\
&+9A_3^2(M+N)^2(L+M)+9B_2^2(N+L)^2(M+L)\\
&+9B_1^2(N+L)^2(N+M)+9C_1^2(L+M)^2(N+M)\\
&+9C_2^2(L+M)^2(N+L)-6A_1A_2N(M+N)^2-6A_1A_3M(M+N)^2\\
&-6B_2B_3L(N+L)^2-6B_2B_1N(N+L)^2-6C_3C_1M(L+M)^2\\
&-6C_3C_2L(L+M)^2-18A_2A_3L(M+N)^2-18B_3B_1M(N+L)^2\\
&-18C_1C_2N(L+M)^2+18A_2B_3(L+N)(MN-H)\\
&+18B_3C_1(M+L)(NL-H)+18C_1A_2(N+M)(LM-H)\\
&-18C_2B_3L(L+M)(L+N)-18A_3C_1M(M+L)(M+N)\\
&-18B_1A_2N(N+L)(N+M)+18C_2A_3(L+M)(MN-H)\\
&+18A_3B_1(M+N)(NL-H)+18B_1C_2(N+L)(LM-H)\\
&+18B_1C_1(M+N)(L^2+H)+18A_2C_2(N+L)(M^2+H)\\
&+18A_3B_3(L+M)(N^2+H)+6A_1B_1(M+N)(H+N^2)\\
&+6A_1C_1(M+N)(H+M^2)+6B_2C_2(N+L)(H+L^2)\\
&+6A_2B_2(N+L)(H+N^2)+6A_3C_3(L+M)(H+M^2)\\
&+6B_2C_3(L+M)(H+L^2)+6A_1B_2[(2N-M)H-MN^2]\\
&+6A_1C_2[(2M-N)H-M^2N]+6B_2C_1[(2L-N)H-L^2N]\\
&+6B_3A_2[(2N-L)H-LN^2]+6A_2C_3[(2M-L)H-LM^2]\\
&+6B_1C_3[(2L-M)H-L^2M]-12D\{A_1(M+N)(H-MN)\cdot
\end{aligned}$$

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$$+B_2(N+L)(H-NL)+C_2(L-M)(H-LM)$$
$$+3A_2M(M+N)(L-N)+3A_3N(M+N)(L+M)$$
$$+3B_3N(N-L)M+L)+3B_1L(N+L)(M+N)$$
$$+3C_1L(L+M)(N-M)+3C_2M(L+M)(N-L)\}$$
$$+36D^2(M+N)(N+L)(L+M)=0.$$

Hence may be deduced the normal form, which represents both the primitive and reciprocal curves,

$$0=\frac{1}{(4\Delta^2)^2}(a_1p^3+b_1q^3+c_1r^3+\ldots)(b_2^2c_3^2P^2+c_3^2a_1^2Q^2+a_1^2b_2^2R^2+\ldots)$$
$$+48d^2(a_1pQ^2R^2+b_2qP^2R^2+c_3rP^2Q^2)+8d^2\{P^2(2Qq+2Rr-Pp)b_2c_3$$
$$+Q^2(2Rr+2Pp-Qq)a_1c_3+3P^2Q(Qq-2Rr)b_1c_3$$
$$+3PQ^2(Pp-2Rr)a_2c_3-6PQ^2ra_2c_3-6P^2Qrb_2c_3-9P^2Q^2rc_1c_3\}$$
$$+12d^2\{PQ^2(2Qq-3Pp)a_1b_2c_3+P^2Q(3Pp-3Qq)a_2b_2c_3$$
$$+PQ^2(Pp-2Rr)a_1c_2^2+P^2Q(Qq-2Rr)b_2c_1^2+Q^3(Qq-4Rr)a_1c_1c_3$$
$$+P^2(Pp-4Rr)b_2c_1c_2-Q^3pa_1a_3c_3-P^2qb_1b_2c_3+Q^2pa_1c_1^2+P^2qb_2c_3^2$$
$$+P^2Q(8Rr-2Pp-Qq)a_2b_2c_3+PQ^2(8Rr-2Qq-Pp)b_1a_2c_3$$
$$+PQ^2(6Rr-3Pp)a_2c_1c_2+P^2Q(6Rr-3Qq)b_1c_1c_3$$
$$+Q^3(Qq-4Pp)a_2a_3c_3+P^2(Pp-4Qq)b_1b_2c_3-2P^2Q^2ra_2b_2c_3$$
$$+P^2rb_2^2c_3+Q^2ra_2^2c_3+3P^2Q^2ra_3c_2^2+3P^2Q^2rb_2c_1^2+6PQ^2ra_2c_1c_3$$
$$+6P^2Qrb_2c_1c_2-P^2rb_2c_2c_3-Q^2ra_1c_1c_3+4P^2Qrb_1c_2c_3$$
$$+4PQ^2ra_2c_1c_3-5P^2Q^2ra_2c_2c_3-5P^2Q^2rb_1c_1c_3+2P^2Qrb_2c_1c_3$$
$$+2PQ^2ra_1c_2c_3\}+6d\{P^2Q(3Qq-2Pp)a_1b_2c_3c_3+4P^3(Rr-Qq)a_2b_2c_3c_3$$
$$-P^2qb_1b_2c_3c_3+P^2Q(Qq+4Rr-2Pp)a_1b_2c_3^2-3P^2qb_1b_2c_3^2$$
$$+4P^2Q(Pp-Qq-Rr)a_1b_2^2c_3+4P^2(Qq-Rr)a_2b_2^2c_3+4P^2qb_1b_2^2c_3$$
$$+PQ^2(Pp-4Qq+10Rr)a_1b_2c_1c_3+3P^2Q(2Pp+Qq-8Rr)a_2b_2c_1c_3$$
$$+3PQ^2(2Rr+2Qq-Pp)b_1b_2c_1c_3-P^2qb_2b_2c_1c_3$$
$$-2PQ^2(Qq-2Pp+Rr)a_1b_1c_2c_3+P^2Q(4Pp-3Qq-6Rr)a_2b_1c_2c_3$$
$$+P^2(6Qq-Pp-2Rr)b_1^2c_2c_3+P^2(Pp-2Qq-2Rr)b_1b_2c_1c_3$$
$$+P^2qb_2^2c_1c_3+4P^2Q(2Rr-Qq)b_1^2c_1c_3+Q^2(6Rr-Qq-2Pp)a_1a_3c_3^2$$
$$+3P^2Q(Qq-2Rr)a_2b_1c_2^2+4P^2(Rr-Qq)a_2b_2c_2^2$$
$$+4Q^2(Pp-Rr)a_1a_2b_2c_3+4PQ^2(2Pp-Qq-Rr)a_2a_2b_2c_3$$

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$$
\begin{aligned}
&- 2P^3Qra_3b_2c_2c_3 - 6P^3Qra_3b_3c_2^2 + 8P^3Qra_3b_3^2c_3 - 6P^2Q^2ra_3b_3c_1c_2 \\
&- 3P^4rb_3^2c_1c_2 + 3P^2Q^2ra_3b_1c_2c_3 - P^4rb_1b_3c_2c_3 - P^2Q^2ra_3b_2c_1c_3 \\
&- P^4rb_2b_3c_1c_3 + 2PQ^3ra_3b_1c_1c_3 - 6PQ^3ra_3^2c_2^2 + P^4rb_1b_2c_3^2 + 4P^4rb_2c_1c_2^2 \\
&- 2P^3Qrb_1^2c_2^2 - 2P^3Qra_2b_2c_3^2 - 12P^3Qrb_1c_1c_2^2 - 4P^3Qrb_2c_1^2c_2 \\
&+ P^2Q^2ra_1b_2c_3^2 + 5P^3Q^2ra_2b_1c_3^2 + 12P^2Q^2b_1c_1^2c_3\} \\
&+ P^3(2Qq + 2Rr - Pp)\,a_1b_2^2c_3^2 - 3P^4qa_2b_2^2c_3^2 - 3P^4ra_3b_2^2c_2^2 \\
&+ 4P^3(2Qq + 2Rr - Pp)\,a_1b_2c_2^3 - 12P^4qa_2b_2c_2^3 - 12P^4ra_3b_2c_2^3 \\
&+ 4P^3(2Qq + 2Rr - Pp)a_1b_3^2c_3 - 12P^4qa_2b_3^2c_3 - 12P^4ra_3b_3^2c_2 \\
&+ 3P^3(Pp - 2Qq - 2Rr)\,a_1c_2^2b_3^2 + 9P^4qa_2c_2^2b_3^2 + 9P^4ra_3b_3^2c_2^2 \\
&+ 6P^3(Pp - 2Qq - 2Rr)\,a_1b_2b_3c_2c_3 + 18P^4qa_2b_2b_3c_2c_3 \\
&+ 18P^4ra_3b_2b_3c_2c_3 - 6P^2Q(Pp - 2Qq + Rr)a_1b_3^2c_1c_2 \\
&- 18P^3(Qq - Rr)\,a_2b_3^2c_1c_3 - 6P^2Q(Pp - Qq - Rr)\,a_1b_2b_3c_1c_3 \\
&- 6P^3(Qq - Rr)\,a_2b_2b_3c_1c_3 - 6P^4qb_1b_2b_3c_1c_3 - 12P^3Qra_3b_2b_3c_1c_3 \\
&+ 6P^2Q(2Qq - 3Pp + 5Rr)\,a_1b_1b_3c_2c_3 - 6P^3(Qq - Rr)\,a_2b_1b_3c_2c_3 \\
&- 12P^4qb_1^2b_3c_2c_3 - 48P^3Qra_3b_1b_3c_2c_3 + 6P^2Q(Pp - Qq - Rr)\,a_1b_1b_2c_3^2 \\
&+ 6P^3(Qq - Rr)\,a_2b_1b_2c_3^2 + 3P^4qb_1^2b_2c_3^2 \\
&+ 12P^3Qra_3b_1b_2c_3^2 - 6P^2Q(Qq - 2Pp + 4Rr)\,a_1b_1c_2^3 \\
&+ 9P^4qb_1^2c_2^3 + 36P^3Qra_3b_1c_2^3 + 6P^3Q(2Pp - 3Qq)\,a_1b_2c_1c_2^2 \\
&+ 24P^3(Qq - Rr)\,a_2b_2c_1c_2^2 + 6P^4qb_1b_2c_1c_2^2 + 12P^3Qra_3b_2c_1c_2^2 \\
&- 6PQ^2(Pp - 2Rr)\,a_1b_1^2c_3^2 - 6P^2Q(Pp - 2Qq + Rr)\,a_2b_1^2c_3^2 \\
&+ P^3(Pp - 8Qq + 4Rr)\,b_1^2c_2^2 - 12P^3Q^2ra_3b_1^2c_2^2 - 3P^4rb_1^2b_3c_2^2 \\
&- 12PQ^2(Pp - 2Rr)\,a_1a_2c_2^3 - 36P^3Q^2ra_2a_3c_2^3 \\
&- 6PQ^2(Pp - Qq + Rr)\,a_1a_2b_2c_3^2 - 6P^2Q(Qq - 2Rr)\,a_2^2b_2c_2^2 \\
&- 6P^3Q^2ra_2a_3b_2c_3^2 - 6PQ^2(3Pp - 2Qq)\,a_1b_1c_1c_2^2 \\
&+ 9P^3(Pp - 4Qq)\,b_1^2c_1c_2^2 - 18P^2Q(2Pp - Qq - 4Rr)\,a_2b_1c_1c_2^2 \\
&- 36P^2Q^2ra_2b_1c_1c_2^2 - 6PQ^2(3Pp - 2Qq)\,a_1b_2c_1^2c_2 \\
&+ 6P^2Q(2Pp - 3Qq)\,a_2b_2c_1^2c_2 - 6P^3(Pp - 4Rr)\,b_1b_2c_1^2c_2 \\
&- 3P^4qb_2^2c_1^2c_2 - 12P^4rb_2b_3c_1^2c_2 - 12PQ^2(Pp - 2Rr)\,a_1a_2b_3^2c_3 \\
&+ 12P^2Q(Qq - 2Rr)\,a_2a_3b_3^2c_3 - 24P^3(Qq - Rr)\,a_2b_1b_3^2c_3 \\
&- 24P^3Q^2ra_2^2b_3^2c_3 + 12PQ^2(Pp - 2Rr)\,a_1b_3^2c_1^2
\end{aligned}
$$

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$$
\begin{aligned}
&- 18P^4Q\,(Pp - 3Rr)\,a_2b_3^2c_1^2 + 9P^3\,(Pp - 4Rr)\,b_1b_2^2c_1^2\\
&+ 3P^4qb_2^2b_3^2c_1^2 + 18P^2Q^2ra_3b_3^2c_1^2 + 9P^4rb_3^3c_1^2\\
&+ 6PQ^2\,(Pp + Qq - 5Rr)\,a_1a_3b_3c_2^2 - 18P^2Q\,(Qq - 2Rr)\,a_2a_3b_3c_2^2\\
&+ 18P^3\,(Qq - 2Rr)\,a_3b_1b_3c_2^2 + 18P^2Q^2ra_3^2b_3c_2^2\\
&+ 6PQ^2\,(Pp - Qq + Rr)\,a_1a_3b_2c_2c_3 + 6P^2Q(Qq - 2Rr)\,a_2a_3b_2c_2c_3\\
&+ 6P^3\,(Qq - Rr)\,a_3b_1b_2c_2c_3 + 6P^2Q^2ra_3^2b_2c_2c_3\\
&+ 6PQ^2\,(Pp - 2Rr)\,a_1b_1b_3c_1c_3 + 24P^2Q\,(Pp - Qq - Rr)\,a_2b_1b_3c_1c_3\\
&- 6P^3\,(Pp - 4Qq)\,b_1^2b_3c_1c_3 + 42P^2Q^2ra_3b_1b_3c_1c_3 + 6P^4rb_1b_3^2c_1c_3\\
&+ 6P^2Q\,(Qq - 2Rr)\,a_2^2b_3c_2c_3 + 6PQ^2\,(3Pp - Qq - 3Rr)\,a_1a_2b_3c_2c_3\\
&+ 6Q^3\,(Pp - Rr)\,a_1a_2b_3c_1c_2 + 42P^2Q^2ra_2a_3b_3c_2c_3\\
&+ 18PQ^2\,(Qq - 2Pp + 3Rr)\,a_2a_3b_3c_1c_2\\
&+ 18P^2Q\,(Pp - 2Qq + 3Rr)\,a_3b_1b_2c_1c_2 - 6Q^3\,(Pp - Rr)\,a_1^2b_3c_3c_2\\
&- 12PQ^2ra_1a_3b_3c_2c_3 + 6PQ^2\,(Qq - 4Pp + 5Rr)\,a_2^2b_3c_1c_3\\
&+ 6Q^3\,(Pp - Rr)\,a_1a_2b_3c_1c_3 - 48PQ^2ra_2a_3b_3c_1c_3 - 12P^3Qra_3b_3^2c_1c_3\\
&+ Q^3\,(2Pp - Qq + 2Rr)\,a_1^2b_2c_3^2 + Q^3\,(Qq + 4Pp - 8Rr)\,a_1^2c_2^3\\
&+ 6Q^3\,(Pp - Rr)\,a_1a_2b_1c_3^2 + 12PQ^2ra_1a_3c_2^3\\
&+ 6PQ^2\,(2Pp - Qq - Rr)\,a_2^2b_1c_3^2 + 12PQ^2ra_2a_3b_1c_2^2\\
&+ 24Q^3\,(Pp - Rr)\,a_1b_1c_1^2c_2 + 18PQ^2\,(Pp - 2Qq + 4Rr)\,a_2b_1c_1^2c_2\\
&+ 36P^2Q\,(Qq - 2Rr)\,b_1^2c_1^2c_2 + 36P^3Qrb_1b_3c_1^2c_2\\
&+ 3c_3\,\{c_1\,(b_2P^2 - 2b_1PQ + a_2Q^2) - c_2\,(a_1Q^2 - 2a_2PQ + b_1P^2)\}^2\,r.
\end{aligned}
$$

This equation to the evolute is incomplete as it stands; the missing terms can be produced (1) by a process of dualising, which is exhibited in the coefficients of d^4, d^3, and d^2, arranged in pairs on this plan of commutation: $\begin{matrix} a_1 \\ a_2 \\ c_1 \end{matrix} \Big| \begin{matrix} b_2 \\ b_1 \\ c_2 \end{matrix}, \begin{matrix} P \\ p \end{matrix} \Big| \begin{matrix} Q \\ q \end{matrix}$, while R, r and c_3 remain fixed, (the principle of this exchange may be observed in the given equation to the cubic curve); (2) by adding terms in the usual cycles a_1, b_2, c_3; a_2, b_3, c_1; a_3, b_1, c_2; P, Q, R; p, q, r; as is shown in the coefficient of d^4.

Cor. Since P, Q, R, or their powers occur in every term, and become zero when $p = q = r$, the evolutes of the cubic curves are parabolic.

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15. To determine the Boothian equation to the evolute of a cubic curve of the third class.

By proceeding as in §§5, 6, the equation to the curve, whether spherical or plane, may be derived from the three-point form given in §14. If $R = 0$, the evolute to the plane curve is at once found, to the extent of half the terms; the other half is obtained by dualising, as explained in §14, (1). (To promote accuracy, the Boothian equation was calculated, first, as a guide to the three-point form).

16. To determine the three-point tangential equation to the evolute of a curve of the third order.

If the variables p, q, r are interchanged with P, Q, R in the processes and equations of §14, the equation to the evolute can be obtained after transforming to tangential coordinates the trilinear equation to the cubic

$$a_1\alpha^3 + 3a_2\alpha^2\beta + \ldots = 0.$$

17. To determine the Boothian tangential equation to the evolute of a curve of the third order.

The required equation may be deduced from the equation of §16, when found, by substitutions explained in §§5, 6 according as a spherical or a plane curve is considered.

18. To determine the duevolute of a cubic of the third order.

By the duevolute has been denoted the locus of the foot of the subtangent, if Cartesian coordinates be used, or the intersection with the tangent of a line drawn through the pole of the circumscribed circle perpendicular to the radius vector from that pole. (*Journal*, Vol. x., p. 321). The duevolute of a spherical curve is the reciprocal curve of the evolute, or the locus of the quadrantal poles of the normal arcs of the curve.

Just as a curve of the third class can be expressed with reference to its three foci $(\mu_1\mu_2\mu_3)$, $[\mu_1\mu_2\mu_3 + (4\Delta^2)\,\mu_4 = 0]$, so, by dualising, a curve of the third order can be expressed in the same form for a spherical curve with reference to its three cyclic arcs. In a plane curve, the reciprocals of the foci are three director-lines, whose doctrine has been explained both geometrically and analytically in the case of conics (*Journal*, Vol. IX., pp. 192, 331). Every cubic of the third order can be expressed in Cartesian coordinates in the form

$$\mu_1\mu_2\mu_3 + (x^2 + y^2)^2\,\mu_4 = 0,$$

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and in trilinear coordinates

$$\mu_1\mu_2\mu_3 + \mu_4\{\Sigma(a^2\alpha^2 + 2bc\beta\gamma\cos 2A)\}^2 = 0.$$

Since then it has been shewn that the evolute of a curve of a third class can be exhibited in the form $u_3v_3 + 16\Delta^4 w_5 = 0$, and since the form of the duevolute is identical for spherical, and nearly identical for plane curves, this equation shews that the cyclic arcs and director-lines of the cubic and its reciprocal are also common to the duevolute. As in a plane curve of the third class the reciprocal curve degenerates into foci, which are cusps on the evolute situated at the line at infinity, whose radii-vectores are perpendicular to the asymptotes (*Journal*, Vol. x., p. 335), so in a plane curve of the third order the reciprocal curve degenerates into director-lines, which are stationary tangents through the origin (which is a common point of inflexion to the evolute); these tangents are perpendicular to the tangents which are drawn to the cubic curve from the origin. This conclusion may be justified by combining the linear equation to a director-line $(\mu_1 = 0 \equiv dx + ey)$ with that to the origin $\{(x^2 + y^2)^2 = 0\}$. The origin is not a point of undulation, but of inflexion, since the stationary tangent passes through four consecutive points, two of which are coincident and correspond to the two consecutive points of contact on the primitive curve lying on the tangent through the origin. As in the case of a parabolic curve of the third class, two cusps unite to form a point of inflexion at infinity on the evolute, so if the curve of third order pass through the origin, two points of inflexion unite at the origin in a cusp on the duevolute.

19. By way of illustration, the points of inflexion and cusp at three different origins of the du-envelope of the same hyperbola are shown in (figs. 28, 29). The du-evolute of a parabola, if its vertex be the origin, is of a cissoidal form, the vertex being consequently the cusp.

Cheltenham, *Sept.* 10, 1870.

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ON THE MOTION OF FLUIDS.

By Sir James Cockle, F.R.S.

(Continued from Vol. x., p. 311.)

67. To the preceding results a yet greater generality, mathematical and hydronamical, can be given. And in these papers I shall extend the processes so as to obtain rigorous solutions of problems of which I believe that no rigorous solution has yet been given, and one of which has hitherto, I think, resisted attempts at rigorous solution. Moreover it will be found that when the density and the pressure are connected by a binomial algebraical (though not necessarily linear) relation, rigorous solutions can be obtained.

68. Designating as universal, particular, and singular the three genera of solutions of a system of differential equations involving more than one dependent variable, by universal solutions, I mean indifferently those which, in the terminology of the preceding paper, are called complete, general, and perfect solutions. By particular solutions, I mean indifferently those therein termed incomplete, generalized, and imperfect solutions. There is also a kind of solution which may partake partly of the nature of a particular solution and partly of the nature of a singular solution, and which may therefore be distinguished as a "mixed" solution. And an inspection of equation (47) of p. 355 of Boole, seems to indicate that there may be a mixed solution of a partial differential equation of the first order involving more than two independent variables.

69. The solutions obtained in Art. 9 and Art. 14, respectively, seem to be essentially different, and to indicate that the theorem at p. 343 of Boole has nothing corresponding to it in the theory of systems of equations involving more than one dependent variable. I propose to call the solution of Art. 9 a mediate solution, because it is effected by a change of independent variable, and to call that of Art. 14 an immediate solution. It will be found that from a mediate solution

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it is not possible to deduce a general or a perfect solution, but that it is possible to obtain, from a mediate solution, a solution of a more general system than that from which it is derived; and this species of generalization I call "inverse generalization."

70. When, in order to generalize a solution, one or more of the arbitrary constants is made an arbitrary function of another, I shall, following Mr. De Morgan (on some points in the Theory of Differential Equations, *C. T.*, Pt. IV. Vol. IX., Art. 2 of the Memoir), call such constants compensating variables. When one or more of the arbitrary constants is or are made a definite function or functions of others, I shall call such constants related parameters. If any constants which enter into the coefficients of a differential equation involving one dependent and one independent variable be treated as compensating variables, then the process of inverse generalization may be applied.

71. We may denote by D differentiation with respect to a compensating variable whereof other compensating variables are, in the differentiation, regarded as arbitrary functions. So that D will in general only apply to the compensating variables. If δv denote any arbitrary infinitesimal increment given, or supposed to be given to v, then δv may be called the variation of v, and δ regarded as the symbol of variation used in the calculus of variations. If dv denote the infinitesimal increment actually taken by v under given conditions, say in the infinitesimal time dt, then dv is the differential of v. We have, always, the known relation $d\delta v = \delta dv$, and dv and δv are independent, and we may suppose δv to be infinitesimally small compared with dv, or *vice versâ*.

72. The rest of the present paper will be divided into four sections. The first will consist of supplementary or emendatory remarks. In the second I shall discuss the mediate solution given in Art. 9 and its inverse generalization. In the third I shall enter upon the subject of Spherical Waves, and in the fourth I shall consider cases in which the pressure varies as a power of the density, cases which include, among others, that in which the pressure follows Laplace's law (as to which, see Thomson and Tait's *Natural Philosophy*, § 824).

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§ 1. *Supplementary Remarks.*

73. The observations made in Art. 24 apply to the investigations in Arts. 40–49, and we may eliminate M from (μ) not only by means of (ι) but also by means of (κ) and (λ). By so doing we shall obtain the additional equations

$$\frac{du}{dx}+n\frac{du}{dy}+(p+nq)\frac{du}{dz}+\frac{V}{S}\left(n\frac{du}{dp}+\frac{du}{dq}\right)=0\ldots(\rho_2),$$

and

$$\frac{du}{dx}+n\frac{du}{dy}+(p+nq)\frac{du}{dz}+\frac{V}{T}n\frac{du}{dq}=0\ldots\ldots\ldots\ldots(\rho_3),$$

and in integrating the latter we may treat p as a constant, remembering always that (ξ_2) must ultimately be satisfied. I have unduly restricted the process of Arts. 40–49, which includes that of Monge, as well as gives results where Monge's method would fail. If the c on the dexter of (γ), instead of being supposed constant, be supposed to be a function satisfying the equation

$$\frac{dc}{dx}+n\frac{dc}{dy}=0\ldots\ldots\ldots\ldots\ldots\ldots(\phi),$$

then (γ) still gives a first integral of (α). This may be shown thus. In Art. 41 u is regarded as a function of x, y, z, p and q. But it may also be regarded as being indirectly a function of x and y only, for z, p, and q are ultimately functions of x and y. Denoting by brackets differentiation on the latter hypothesis, then, after determining u by the given method, we can derive from (γ) the relation

$$\left(\frac{du}{dx}\right)+n\left(\frac{du}{dy}\right)=M(Rr+Ss+Tt-V)\ldots\ldots(\chi),$$

on the supposition that c is constant. But the same relation may be derived from (γ) when c is variable, provided only that it satisfy (ϕ), which will be the case if c be an arbitrary function of v, the definition of v being that $v=a$ is the solution of $dy=ndx$. The true enunciation of the result of the method of Arts. 40–49 is therefore as follows. When (ξ_2) is satisfied, then

$$u=f(v)\ldots\ldots\ldots\ldots\ldots\ldots(\psi),$$

wherein f is arbitrary is a first integral of (α). When (ξ_2) is not satisfied, but a relation of the form

$$\frac{du}{dq}-m\frac{du}{dp}=U_2\left(\frac{dU_1}{dq}-m\frac{dU_1}{dp}\right)\ldots\ldots\ldots(\omega)$$

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holds, then $U_2 = 0$ ..... (ψ_2)

is a first integral of (a).

74. The concluding sentence of Art. 30 should be expunged, and in Art. 31 the word "latter" should replace "former," and *vice versâ*. The argument of Art. 37 should be amended thus:

Case I. Let (K) be integrated in the form (L). Then (L) is the condition that (v) and (w) may have a common solution. By means of (L) eliminate, say a_4 from (v), (w), (x), and (y), and call the respective results

$$V_1 = 0 \text{ ..... } (N),$$
$$V_2 = 0 \text{ ..... } (O),$$
$$Z_1 = 0 \text{ ..... } (M_1),$$
$$Z_2 = 0 \text{ ..... } (M_2).$$

Then (N) and (O) have a common solution. Call this common solution

$$W = 0 \text{ ..... } (U).$$

In $Z_1 = 0$, $Z_2 = 0$, and $W = 0$ replace a_2 by $\phi(a_1)$ and a_3 by $\chi(a_1)$ and call the respective results $(Z_1) = 0$, $(Z_2) = 0$, and $(W) = 0$. The elimination of a_1 between $(Z_1) = 0$ and $(W) = 0$ will give a generalized implicit value of z_1, and a like value of z_2 will be obtained by eliminating between $(Z_2) = 0$ and $(W) = 0$. Cases II., III., and IV. require a similar correction, but I do not go into details, because the best practical process is one imperfectly sketched in Art. 39, and hereafter actually employed.

75. With respect to Art. 29, and to functional symbols, I would refer to Art. 19 of Mr. De Morgan's paper already cited, and with respect to the extent of arbitrariness which a function may possess I would cite the concluding paragraph of Section IV. of his paper "On some points of the Integral Calculus," in the *Cambridge Transactions* (Vol. IX., Part II).

76. Boole observes (p. 350, Art. 12) that the general and the complete solutions of a certain equation are manifestly equivalent. I am not aware that there is any known process for passing from a general or generalized to a perfect or imperfect solution. But from the generalized solution at

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the end of Art. 54 of this set of papers we may deduce an imperfect (functional) solution as follows.

77. Put [17] and [24] under the respective forms

$$(t+m)(u \pm A) = x - \phi(m) \ldots\ldots\ldots\ldots [57],$$

$$(t+m)\frac{d\phi(m)}{dm} = x - \phi(m) \ldots\ldots\ldots\ldots [58],$$

and subtract [58] from [57]. We have

$$(t+m)\left\{u \pm A - \frac{d\phi(m)}{dm}\right\} = 0,$$

or

$$u \pm A - \frac{d\phi(m)}{dm} = 0 \ldots\ldots\ldots\ldots\ldots [59].$$

Now we may write [59] in the form

$$u \pm A - \psi^{-1}(m) = 0 \ldots\ldots\ldots\ldots [59_2],$$

whence

$$m = \psi(u \pm A) \ldots\ldots\ldots\ldots\ldots [60],$$

and from [57] we obtain

$$t - \frac{x}{u \pm A} = -\frac{\phi\psi(u \pm A)}{u \pm A} - \psi(u \pm A) \ldots\ldots [61],$$

and, as remarks of Mr. De Morgan show, the dexter of [61] is an arbitrary function of $u \pm A$ and therefore of u. Hence we may deduce [47] without having recourse to a differential equation of the second order.

78. The method employed in these papers for the solution of problems of fluid motion may be summarized in the following rule.

RULE. In order to solve a system of two differential equations of the first order involving two dependent and two independent variables, assign at pleasure one of the dependent variables as a function of the independent variables; substitute the assigned value in one of the equations and solve for the other dependent variable; in the solution determine the arbitrary function, or constants, as the case may be, in such manner, when practicable, as to satisfy the second equation; then generalize the resulting solution.

SCHOLIUM. The solutions already arrived at may be at once obtained by assuming (13) to hold and proceeding as

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in the Rule. The advantage of this assumption is that, by it, we may express the place in terms of the time. Other assumptions, leading to solutions more or less incomplete, may however be readily found. This is the case if we make the velocity a function of the time only, or of the place only. If all known modes of solving a partial biordinal fail, we may seek a complete or incomplete solution, and then endeavour to generalize it. In Mr. De Morgan's paper, first above mentioned, Art. 17, the generalization of a primary [*i.e.* complete] solution of a biordinal is briefly considered.

79. The notation hereinbefore used might, in some instances, be improved, but I adhere to it for the sake of greater facility in referring to and comparing foregoing results.

§ 2. *The Mediate Solution and its Generalization.*

80. Recurring now to Art. 9 we obtain, from (47) and (66),

$$\xi = \frac{1}{v}\{x + C(Dt + E)\} - \frac{2C}{B^2} \quad \ldots\ldots\ldots (124).$$

Hence (65) becomes

$$u = \left[\xi + \frac{2C}{B^2} - \frac{CD}{B\sqrt{\{\log(mv)\}}}\right] B\sqrt{\{\log(mv)\}}$$

$$= \left[B\xi + \frac{2C}{B}\right]\sqrt{\{\log(mv)\}} - CD \quad \ldots\ldots\ldots (125).$$

81. We also obtain, from (61*b*),

$$\log\frac{n}{m}\rho = -\frac{1}{4A^2}\{B^2\xi^2 + 4C\xi\} - \log(mv)$$

$$= -\frac{1}{4A^2}\left\{\left(B\xi + \frac{2C}{B}\right)^2 - \left(\frac{2C}{B}\right)^2\right\} - \log(mv)$$

$$= -\frac{1}{4A^2}\left\{\frac{(u+CD)^2}{\log(mv)} - \frac{4C^2}{B^2}\right\} - \log(mv) \quad \ldots\ldots (126).$$

82. Now if, in conformity with Art. 7, we change the independent variable t, and if, moreover, we put

$$\log\left(\frac{n\rho}{m}\right) = r \quad \ldots\ldots\ldots [19],$$

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then the equations (123) and (2) respectively become

$$B\sqrt{\{\log(mv)\}}\frac{du}{dv}+u\frac{du}{dx}+A^2\frac{dr}{dx}=0 \ldots\ldots (127),$$

$$B\sqrt{\{\log(mv)\}}\frac{dr}{dv}+u\frac{dr}{dx}+\frac{du}{dx}=0 \ldots\ldots\ldots (128),$$

and of this system equation (65) and Arts. 80–82 afford us the following solution:

$$u=\frac{B\sqrt{\{\log(mv)\}}}{v}\{x+C(Dt+E)\}-CD \ldots (129),$$

$$r=-\frac{1}{4A^2}\left\{\frac{(u+CD)^2}{\log(mv)}-\frac{4C^2}{B^2}\right\}-\log(mv) \ldots (130).$$

83. A more convenient form can be given to this solution by putting

$$CD=a,\ CE=\frac{b}{B},\ \frac{4C^2}{B^2}=c^2 \ldots\ldots\ldots (131),$$

and eliminating t by means of (58). We can thus write the solution in the form

$$u=\frac{\sqrt{\{\log(mv)\}}}{v}\left[Bx+a\int\frac{dv}{\sqrt{\{\log(mv)\}}}+b\right]-a \ldots (132),$$

$$r=-\frac{1}{4A^2}\left\{\frac{(u+a)^2}{\log(mv)}-c^2\right\}-\log(mv) \ldots\ldots\ldots\ldots (133).$$

84. Next, adopting the notation of Art. 71, we see that, in order that this solution may be generalized, Du and Dr must have a common factor. Now Du will be this factor provided that a, c, and m be treated simply as arbitrary constants and B and b alone be treated as compensating variables. On this supposition we have

$$Dr=-\frac{1}{2A^2}\cdot\frac{u+a}{\log(mv)}\cdot Du \ldots\ldots\ldots (134),$$

and $Du=0$, the equation of compensation, is seen to be equivalent to

$$xDB+Db=0 \ldots\ldots\ldots\ldots (135),$$

wherein b is to be made an arbitrary function of B. Accordingly assume

$$b=-\int\phi^{-1}(B)\,DB \ldots\ldots\ldots\ldots (136),$$

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then, differentiating with the symbol D, to which the $\int$ is inverse, and dividing out DB, equation (135) gives

$$x - \phi^{-1}(B) = 0,$$

or

$$B = \phi(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (137),$$

where ϕ is an arbitrary function. Moreover we may now write (136) in the form

$$b = -\int \phi^{-1}\phi(x)\, d\phi(x) = -\int x d\phi(x)$$
$$= -x\phi(x) + \int \phi(x)\, dx \quad \ldots\ldots\ldots\ldots\ldots (138).$$

85. Hence substituting in (132) the values of B and b respectively given by (137) and (138), and putting

$$\int \phi(x)\, dx = \psi(x) \quad \ldots\ldots\ldots\ldots\ldots (139),$$

where also ψ is an arbitrary function, we find, as the generalized solution

$$u = \frac{\sqrt{\{\log(mv)\}}}{v}\left[\psi(x) + a\int \frac{dv}{\sqrt{\{\log(mv)\}}}\right] - a \ldots (140),$$

and the generalized value of r will be found by substituting for u in (133) the value given by (140).

86. But, inasmuch as one of the compensating variables, viz. B, itself enters into the system, (127) and (128), of Art. 82, the generalization is inverse, that is to say, it gives us, not a generalized solution of the system [20] and [21] of Art. 54, but a solution of a generalized form of the system (127) and (128), to wit, a solution of the system

$$\frac{d\psi(x)}{dx}\sqrt{\{\log(mv)\}}\frac{du}{dv} + u\frac{du}{dx} + A^2\frac{dr}{dx} = 0 \ldots (141),$$

$$\frac{d\psi(x)}{dx}\sqrt{\{\log(mv)\}}\frac{dr}{dv} + u\frac{dr}{dx} + \frac{du}{dx} = 0 \ldots\ldots (142),$$

wherein $\dfrac{d\psi(x)}{dx}$, which replaces B, may be any function whatever of x.

87. I shall verify this inverse generalization, writing for the purpose

$$\psi(x) = \psi, \quad \frac{d\psi(x)}{dx} = \psi_1, \quad \frac{\sqrt{\{\log(mv)\}}}{v} = V \ldots (143).$$

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Now from (140) we have

$$\frac{du}{dv} = \frac{dV}{dv}\left[\psi + a\int\frac{dv}{\{\log(mv)\}}\right] + \frac{a}{v}$$

$$= \frac{1}{V}\cdot\frac{dV}{dv}\cdot(u+a) + \frac{a}{v} \quad \ldots\ldots\ldots\ldots\ldots\ldots (144);$$

therefore $\dfrac{d\psi(x)}{dx}\sqrt{\{\log(mv)\}}\,\dfrac{du}{dv}$

$$= \frac{\sqrt{\{\log(mv)\}}}{V}\,\frac{dV}{dv}(u+a)\,\psi_1 + \frac{a\sqrt{\{\log(mv)\}}}{v}\,\psi_1 \quad \ldots\ldots (145),$$

$$u\,\frac{du}{dx} = Vu\psi_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (146),$$

$$A^2\,\frac{dr}{dx} = -\frac{1}{2\log(mv)}(u+a)\,\frac{du}{dx}$$

$$= -\frac{V}{2\log(mv)}(u+a)\,\psi_1 \ldots\ldots\ldots (147).$$

Hence adding (145), (146) and (147), and remembering that, in virtue of (141), the sinister of the result vanishes, we have

$$0 = \left[\frac{\sqrt{\{\log(mv)\}}}{V}\,\frac{dV}{dv} - \frac{V}{2\log(mv)}\right](u+a)\,\psi_1$$

$$+\left[Vu + \frac{a\sqrt{\{\log(mv)\}}}{v}\right]\psi_1$$

$$= \left[\frac{\sqrt{\{\log(mv)\}}}{V}\,\frac{dV}{dv} + V\left\{1 - \frac{1}{2\log(mv)}\right\}\right](u+a)\,\psi_1$$

$$= \left[v\,\frac{dV}{dv} + \frac{\sqrt{\{\log(mv)\}}}{v} - \frac{1}{2v\sqrt{\{\log(mv)\}}}\right](u+a)\,\psi_1 \ldots (148).$$

But the last of the equations (143) gives

$$\frac{dV}{dv} = -\frac{\sqrt{\{\log(mv)\}}}{v^2} + \frac{1}{2v^2\sqrt{\{\log(mv)\}}} \ldots. (149).$$

Consequently the dexter of (148) vanishes.

88. Again, employing (149) we deduce, from (144),

$$\frac{du}{dv} = \left\{-\frac{1}{v} + \frac{1}{2v\log(mv)}\right\}(u+a) + \frac{a}{v} \ldots. (150),$$

whence

$$2\,\frac{du}{dv} - \frac{u+a}{v\log(mv)} = -2\,\frac{u}{v} \quad \ldots\ldots\ldots (151).$$

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Now, by Art. 85,

$$\frac{dr}{dv} = -\frac{1}{4A^2}\left\{2\frac{du}{dv} - \frac{u+a}{v\log(mv)}\right\}\left\{\frac{u+a}{\log(mv)}\right\} - \frac{1}{v}$$

$$= \frac{1}{2A^2}\left\{\frac{u(u+a)}{v\log(mv)}\right\} - \frac{1}{v} \ldots\ldots\ldots\ldots\ldots\ldots (152),$$

as we see from (151). Hence

$$\frac{d\psi(x)}{dx}\sqrt{\{\log(mv)\}}\frac{dr}{dv} = \frac{\psi_1}{2A^2}\left[\frac{u(u+a)}{v\sqrt{\{\log(mv)\}}}\right] - V\psi_1 \ldots (153),$$

also $$u\frac{dr}{dx} = -\frac{u}{2A^2}\left\{\frac{u+a}{\log(mv)}\right\}\frac{du}{dx} = -\frac{V\psi_1}{2A^2}\left\{\frac{u(u+a)}{\log(mv)}\right\}$$

$$= -\frac{\psi_1}{2A^2}\cdot\frac{1}{v\sqrt{\{\log(mv)\}}}u(u+a) \ldots\ldots\ldots (154),$$

and $$\frac{du}{dx} = V\psi_1 \ldots\ldots\ldots\ldots\ldots (155).$$

Adding the last three equations, and remembering that, in virtue of (142), the sinister of the result vanishes, we have the dexter vanishing identically, and, consequently, we have obtained a complete verification of the inverse generalization.

§ 3. *On Spherical Waves.*

89. I shall now give a rigorous solution of the following case of the problem of spherical waves:

An elastic fluid is in motion under the influence of two forces emanating from the origin of coordinates. One force varies directly and the other inversely as the distance from the origin. Initially, and therefore perpetually, the densities and velocities of all portions of the fluid equidistant from the origin are equal, and the directions of the motions of such portions are similar, that is to say, such motions are either all from, or all towards, the origin. It is required to assign the motion rigorously.

90. Let x be the distance from the origin of a particle whereof the velocity is u and the density σ. Then, p being the pressure, the dynamical equation is

$$\frac{du}{dt} + u\frac{du}{dx} + \frac{1}{\sigma}\frac{dp}{dx} + Æx + \frac{Œ}{x} = 0 \ldots\ldots (1)_3.$$

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91. Conceive two concentric spheres to be described round the origin as centre with the respective radii x and $x+\delta x$. Let the variation δx be infinitesimal compared with x. Then the volume contained between the two spherical surfaces is $4\pi x^2\delta x$, and the mass so contained is $4\pi\sigma x^2\delta x$. And if for convenience we introduce a new variable ρ, connected with the density σ by the relation

$$\rho = \sigma x^2 \quad \text{..... (i)},$$

such mass may be represented by $4\pi\rho\delta x$. Now by the principle of continuity this mass will be invariable, and the invariability will be expressed by

$$d(\rho\delta x) = 0 \quad \text{..... (ii)},$$

where d operates on any oper and f so as to give it an increment df in the elementary time dt. From (ii) we deduce

$$\begin{aligned} 0 &= d\rho\,\delta x + \rho d\delta x = d\rho\,\delta x + \rho\delta dx \\ &= \left(d\rho + \rho\frac{\delta dx}{\delta x}\right)\delta x = \left\{d\rho + \rho\frac{\delta(udt)}{\delta x}\right\}\delta x \\ &= \left(\frac{d\rho}{dt} + u\frac{d\rho}{dx} + \rho\frac{\delta u}{\delta x}\right)dt\,\delta x \\ &= \left(\frac{d\rho}{dt} + u\frac{d\rho}{dx} + \rho\frac{du}{dx}\right)dt\,\delta x \quad \text{..... (iii)}, \end{aligned}$$

consequently the equation of continuity is

$$\frac{d\rho}{dt} + u\frac{d\rho}{dx} + \rho\frac{du}{dx} = 0 \quad \text{..... (2)},$$

and this and the equation

$$dx - udt = 0 \quad \text{..... (3)}$$

employed in its deduction, are the same in form as the equations of continuity (2) and of definition (3) of Art. 1, and, as before, we shall take

$$p = A^2 \times (\text{density}) \quad \text{..... (4)},$$

which, in this case of spherical waves, becomes, in virtue of (1),

$$p = A^2\sigma = A^2\frac{\rho}{x^2} \quad \text{..... } (4)_2.$$

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92. The equation last preceding gives

$$\frac{dp}{dx} = A^2\left(\frac{1}{x^2}\frac{d\rho}{dx} - \frac{2\rho}{x^3}\right),$$

whence we have

$$\frac{1}{\sigma}\frac{dp}{dx} = \frac{x^2}{\rho}\frac{dp}{dx} = A^2\left(\frac{1}{\rho}\frac{d\rho}{dx} - \frac{2}{x}\right) \ldots\ldots\ldots \text{(iv)},$$

consequently $(1)_2$ becomes

$$\frac{du}{dt} + u\frac{du}{dx} + \frac{A^2}{\rho}\frac{d\rho}{dx} + \text{Æ}x + \frac{\text{Œ} - 2A^2}{x} = 0 \ldots (1)_3,$$

and we have to solve the system constituted by (2) and $(1)_3$. This system shows that the problems of spherical and of plane waves are in a certain sense convertible.

93. Much of the necessary work is already done. Applying the rule given in Art. 78 assume, as in Art. 2,

$$u = \theta x + \Theta \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

then (2) becomes $\frac{d\rho}{dt} + (\theta x + \Theta)\frac{d\rho}{dx} + \theta\rho \ldots\ldots\ldots\ldots$ (v),

whereof the solution, already obtained by a less convenient process in Art. 3, is

$$\rho = e^{-\int\theta dt} f\left(\frac{x+T}{e^{\int\theta dt}}\right) \ldots\ldots\ldots\ldots\ldots (25),$$

or, putting as in Art. 6,

$$e^{\int\theta dt} = v \ldots\ldots\ldots\ldots\ldots\ldots (42),$$

$$\rho = \frac{1}{v} f\left(\frac{x+T}{v}\right) \ldots\ldots\ldots\ldots\ldots (\text{vi}),$$

wherein the function f is arbitrary. The symbols θ and Θ are, as yet, arbitrary functions of the time t only, and T and its properties are sufficiently defined and given in Art. 4. We may here conveniently repeat the equation

$$z = f\left(\frac{x+T}{e^{\int\theta dt}}\right) \ldots\ldots\ldots\ldots\ldots (23),$$

and the equation

$$\xi = \frac{x+T}{e^{\int\theta dt}} \ldots\ldots\ldots\ldots\ldots (35),$$

of Art. 5, which, combined give

$$z = f(\xi) \ldots\ldots\ldots\ldots\ldots (118)_2,$$

as, among other relations, is noticed in Art. 21.

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94. Now, following a path analogous to that pursued in Art. 5, we deal as follows with the dynamical equation marked $(1)_s$:

$$\frac{du}{dt} = \frac{d\theta}{dt}\,x + \frac{d\Theta}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots (30)_s,$$

$$u\frac{du}{dx} = \theta\,(\theta x + \Theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (31)_s,$$

$$\frac{A^2}{\rho}\frac{d\rho}{dx} = \frac{A^2 e^{-\int\theta dt}}{f(\xi)} \cdot \frac{df(\xi)}{d\xi} \quad \ldots\ldots\ldots\ldots\ldots (\text{vii}),$$

$$Ex = E\,(e^{\int\theta dt}\xi - T) \quad \ldots\ldots\ldots\ldots\ldots (\text{viii}),$$

$$\frac{E - 2A^2}{x} = \frac{E - 2A^2}{e^{\int\theta dt}\xi - T} \quad \ldots\ldots\ldots\ldots\ldots (\text{ix}).$$

Hence, adding, we see that $(1)_s$ is equivalent to

$$\left(\frac{d\theta}{dt} + \theta^2\right)(e^{\int\theta dt}\xi - T) + \frac{d\Theta}{dt} + \theta\Theta$$

$$+ \frac{A^2 e^{-\int\theta dt}}{f(\xi)}\,\frac{df(\xi)}{d\xi}$$

$$+ E\,(e^{\int\theta dt}\xi - T) + \frac{E - 2A^2}{e^{\int\theta dt}\xi - T} = 0 \quad \ldots\ldots\ldots\ldots\ldots (\text{x}).$$

95. By means of (29) and (42) respectively eliminate Θ and θ from (x), which then becomes

$$\left(\frac{d^2v}{dt^2} + Ev\right)\xi - \left(\frac{d^2T}{dt^2} + ET\right)$$

$$+ \frac{E - 2A^2}{v\xi - T} + \frac{A^2}{v}\,\frac{1}{f(\xi)}\,\frac{df(\xi)}{d\xi} = 0 \quad \ldots\ldots\ldots\ldots\ldots (\text{xi}).$$

96. Next, multiply the equation last preceding into $vd\xi$, and it becomes

$$\left(v\frac{d^2v}{dt^2} + Ev^2\right)\xi d\xi - \left(\frac{d^2T}{dt^2} + ET\right)vd\xi$$

$$+ \frac{E - 2A^2}{\xi - \dfrac{T}{v}}\,d\xi + \frac{A^2}{f(\xi)}\,\frac{df(\xi)}{d\xi}\,d\xi = 0 \quad \ldots\ldots\ldots\ldots\ldots (\text{xii}),$$

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which will have its sinister a perfect or total differential with respect to ξ, provided that

$$v\frac{d^2v}{dt^2} + Æv^2 = \frac{B^2}{2} \quad \ldots\ldots\ldots\ldots \text{(xiii)},$$

$$v\left(\frac{d^2T}{dt^2} + ÆT\right) = -C \quad \ldots\ldots\ldots \text{(xiv)},$$

and

$$\frac{T}{v} = G \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(xv)},$$

wherein B, C and G are constants.

97. These three equations may be satisfied. For, substituting in (xiv) for T its value Gv as given by (xv), equation (xiv) becomes

$$Gv\left(\frac{d^2v}{dt^2} + Æv\right) = G\left(v\frac{d^2v}{dt^2} + Æv^2\right) = -C \ldots \text{(xvi)},$$

which coincides with (xiii) provided that

$$-C = \frac{GB^2}{2}, \text{ or } GB^2 + 2C = 0 \ldots\ldots\ldots \text{(xvii)}.$$

98. Let this relation subsist among the constants. Multiply (xiii) into $\frac{1}{v}\frac{dv}{dt}$, and it becomes

$$\frac{dv}{dt}\cdot\frac{d^2v}{dt^2} + Æv\frac{dv}{dt} = \frac{B^2}{2}\frac{1}{v}\frac{dv}{dt} \ldots\ldots\ldots \text{(xviii)},$$

whence, integrating, reducing, and transposing,

$$\left(\frac{dv}{dt}\right)^2 = B^2\log(mv) - Æv^2 \ldots\ldots\ldots\ldots (55)_2,$$

whence, again,

$$\frac{dt}{dv} = \frac{1}{\sqrt{\{B^2\log(mv) - Æv^2\}}} \ldots\ldots\ldots\ldots (57)_2,$$

$$t = \int\frac{dv}{\sqrt{\{B^2\log(mv) - Æv^2\}}} \quad \ldots\ldots\ldots (58)_2,$$

and the independent variable is no longer the time t, but the auxiliary quantity v.

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99. In virtue of (xiii), (xiv), (xv), and (xvii) equation (xii) becomes

$$\frac{B^2}{2}\,\xi d\xi - \frac{GB^2}{2}\,d\xi + \frac{\mathcal{E}-2A^2}{\xi-G}\,d\xi + \frac{A^2}{f(\xi)}\,\frac{df(\xi)}{d\xi}\,d\xi$$

$$= \frac{B^2}{2}\,(\xi-G)\,d\xi + \frac{\mathcal{E}-2A^2}{\xi-G}\,d\xi + \frac{A^2}{f(\xi)}\,\frac{df(\xi)}{d\xi}\,d\xi = 0 \ldots (\text{xix}),$$

whence, by integration, we deduce

$$\left\{\frac{B(\xi-G)}{2}\right\}^2 + (\mathcal{E}-2A^2)\log\{B(\xi-G)\} + A^2\log\{nf(\xi)\} = 0 \quad \ldots\ldots\ldots\ldots (60)_2,$$

whence, again, by (118) and (i),

$$A^2\log\{nf(\xi)\} = A^2\log(nv\rho) = A^2\log(nv\sigma x^2)$$

$$= -\left\{\frac{B(\xi-G)}{2}\right\}^2 - (\mathcal{E}-2A^2)\log\{B(\xi-G)\} \ldots\ldots (61)_2.$$

100. Combining (35), (42), and (xv) we see that

$$\xi - G = \frac{x}{v} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\text{xx}),$$

and consequently that $(61)_2$ becomes

$$A^2\log(nv\sigma x^2) = -\left(\frac{Bx}{2v}\right)^2 - (\mathcal{E}-2A^2)\log\left(\frac{Bx}{v}\right),$$

which gives

$$A^2\log\sigma = -\left(\frac{Bx}{2v}\right)^2 - (\mathcal{E}-2A^2)\log\left(\frac{Bx}{v}\right) - A^2\log(nvx^2) \quad \ldots\ldots\ldots\ldots (61b)_2,$$

which determines σ, the density.

101. In order to obtain u, the velocity, we have recourse to (52) and $(57)_2$, and find

$$\theta = \frac{1}{v}\,\frac{dv}{dt} = \frac{\sqrt{\{B^2\log(mv) - \mathcal{E}v^2\}}}{v} \ldots\ldots (63)_2,$$

and, combining (50) and (xv), we have

$$\Theta = \theta T - \frac{dT}{dt} = Gv\theta - G\,\frac{dv}{dt} = 0 \ldots\ldots (64)_2,$$

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for from (52) we learn that $v\theta - \frac{dv}{dt}$ vanishes. Hence, by (13), $u = \theta x$, or

$$u = \sqrt{[\{B^2 \log(mv) - Æv^2\}]} \cdot \frac{x}{v} \ldots\ldots\ldots (65)_2,$$

which determines the velocity u, and shows that for a given value of v, that is to say, for any given time, the velocities are proportional to the distances from the centre. The time is obtained from equation $(58)_2$. The solution is a mediate one, and does not, therefore, admit of direct generalization. Nor in this case does it, I think, admit of an inverse generalization.

102. Let $Æ$ vanish, in other words, let there be no central force varying directly as the distance. Then $(58)_2$ and $(63)_2$ respectively bring us back to

$$t = \frac{1}{B} \int \frac{dv}{\sqrt{\{\log(mv)\}}} \ldots\ldots\ldots\ldots (62),$$

$$\theta = \frac{B}{v} \sqrt{\{\log(mv)\}} \ldots\ldots\ldots\ldots (63),$$

and we have

$$\Theta = 0 \ldots\ldots\ldots\ldots (64)_2,$$

$$u = \frac{Bx}{v} \sqrt{\{\log(mv)\}} \ldots\ldots\ldots\ldots (\text{xxi}),$$

$$T = Gv \ldots\ldots\ldots\ldots (\text{xv})_2,$$

$$\xi = \frac{x}{v} + G = \frac{x + Gv}{v} \ldots\ldots\ldots\ldots (\text{xx})_2,$$

$$\log \sigma = -\left(\frac{Bx}{2Av}\right)^2 - \frac{Œ - 2A^2}{A^2} \log\left(\frac{Bx}{v}\right) - \log(nvx^2) \ldots (61b)_3.$$

103. There is a plane disturbance corresponding in some points with that of the last article. If in Art. 6 we suppose that C vanishes the connection between (43) and (44) is destroyed, and, v being assigned as before, we now have

$$T = Dt + E \ldots\ldots\ldots\ldots (47)_2,$$

$$\Theta = (Dt + E)\,\theta - D \ldots\ldots\ldots\ldots (53)_2,$$

and modifications to meet this case must be made in the

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formulæ of Art. 9. Equations (62) and (63) will stand as before, but we shall now have

$$\Theta = \frac{B(Dt+E)\sqrt{\{\log(mv)\}}}{v} - D \ldots\ldots (64)_2,$$

$$T = Dt + E \ldots\ldots (47)_2,$$

$$\xi = \frac{x+T}{v} = \frac{x+Dt+E}{v} \ldots\ldots (66)_2,$$

$$u = \left[\frac{x+Dt+E}{v}\right] B\sqrt{\{\log(mv)\}} - D \ldots\ldots (65)_2,$$

all which four equations are deducible from the corresponding equations in Art. 9 by respectively therein writing unity in place of C. But in $(61b)$ we must replace C by zero. This being done we have

$$\log\rho = -\left[\left\{\frac{B(x+Dt+E)}{2Av}\right\}^2 + \log(nv)\right] \ldots (61b)_4,$$

and thus will be modified the formulæ of Art. 9 when C is supposed to vanish.

104. Now if, in the formulæ for plane disturbance given in the last article, we make D and E each to vanish, the value of u will be the same as in the case of spherical disturbance considered in Art. 102. And in either case we shall also have Θ vanishing and the values of t and θ coincident.

105. We may give to equation $(61b)_2$ of Art. 100 various forms. Since

$$\log\left(\frac{Bx}{v}\right) = \log B + \log x - \log v,$$

and $$\log(nvx^2) = \log n + 2\log x + \log v,$$

consequently $(\mathfrak{E} - 2A^2)\log\left(\frac{Bx}{v}\right) + A^2\log(nvx^2)$

$$= (\mathfrak{E} - 2A^2)\log B + A^2\log n + (3A^2 - \mathfrak{E})\log v + \mathfrak{E}\log x$$

$$= (\mathfrak{E} - 2A^2)\log(nB) + (3A^2 - \mathfrak{E})\log(nv) + \mathfrak{E}\log x.$$

Call this expression

$$a\log(nB) + b\log(nv) + \mathfrak{E}\log x,$$

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then $(61b)_2$ gives

$$A^2 \log \sigma = -\left(\frac{Bx}{2v}\right)^2 - \{a \log(nB) + b \log(nv) + Œ \log x\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (61b)_3,$$

and if $Œ$ be zero, then the last equation may be put under the form

$$\log \sigma = -\left(\frac{Bx}{2Av}\right)^2 + 2 \log(nB) - 3 \log(nv) \ldots (61b)_6,$$

and when, in $(61b)_4$, D and E are supposed to vanish, the result presents a close resemblance to $(61b)_6$, and to this extent the cases of the plane and of the spherical waves correspond.

When in equation $(1)_2$, we have

$$Æ = 0 \text{ and } Œ - 2A^2 = 0 \ldots\ldots\ldots\ldots (\text{xxii}),$$

then $(1)_2$ coincides with (1) and we are led to a case in which the problem of the spherical disturbance is completely reduced to one of plane disturbance, viz. to that one already so fully discussed. If in the plane solutions we replace ρ by σx^2 we shall have the corresponding spherical solutions.

§ 4. *Laws of Pressure. Laplace's Law.*

106. Resuming the problem of Art. 1, suppose that, in place of (4), we have the relation

$$p = A^2 \rho^n + A_0 \ldots\ldots\ldots\ldots\ldots\ldots (4)_2,$$

in other words, that the pressure is the sum of a multiple of a power of the density added to a constant. In the case of an elastic fluid the constant, A_0, must be taken to vanish. But this is from a physical and not a mathematical necessity, for our formulæ would apply even if A_0 did not vanish.

107. Equations (1), (2), and (3) will stand as before, but the dynamical equation (1) will become, after substitution,

$$\frac{du}{dt} + u \frac{du}{dx} + \frac{A^2}{\rho} \frac{d(\rho^n)}{dx} = 0 \ldots\ldots\ldots (123)_2,$$

$$\frac{du}{dt} + u \frac{du}{dx} + nA^2 \rho^{n-2} \frac{d\rho}{dx} = 0 \ldots\ldots\ldots (\text{xxiii}).$$

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Applying the rule of Art. 78, assume, as in Art. 2,

$$u = \theta x + \Theta \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13),$$

then, as in Art. 93, we deduce from (2)

$$\rho = \frac{1}{v} f\left(\frac{x+T}{v}\right) = \frac{f}{v} \quad \ldots\ldots\ldots \quad (\text{xxiv}),$$

whence

$$\frac{d\rho}{dx} = \frac{1}{v}\frac{df}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\text{xxv}),$$

consequently

$$\rho^{n-2}\frac{d\rho}{dx} = \frac{f^{n-2}}{v^{n-1}}\frac{df}{dx} \quad \ldots\ldots\ldots\ldots (\text{xxvi}),$$

and (xxiii) becomes

$$\frac{du}{dt} + u\frac{du}{dx} + nA^2\frac{f^{n-2}}{v^{n-1}}\frac{df}{dx} = 0 \quad \ldots\ldots\ldots (\text{xxvii}),$$

or, substituting for u its value as given by (13),

$$\left(\frac{d\theta}{dt} + \theta^2\right)x + \frac{d\Theta}{dt} + \theta\Theta + nA^2\frac{f^{n-2}}{v^{n-1}}\frac{df}{dx} = 0 \ldots (\text{xxviii}),$$

but, having regard to (12), this last equation becomes

$$\frac{x}{v}\frac{d^2v}{dt^2} + \frac{d\Theta}{dt} + \theta\Theta + nA^2\frac{f^{n-2}}{v^{n-1}}\frac{df}{dx} = 0 \ \ldots \ (\text{xxix}),$$

and in like manner (29) becomes

$$\frac{d^2T}{dt^2} = \frac{T}{v}\frac{d^2v}{dt^2} - \frac{d\Theta}{dt} - \theta\Theta \quad \ldots\ldots\ldots \quad (29).$$

Hence, combining (xxix) and $(29)_2$, we have

$$\frac{x+T}{v}\frac{d^2v}{dt^2} - \frac{d^2T}{dt^2} + nA^2\frac{f^{n-2}}{v^{n-1}}\frac{df}{dx} = 0 \ \ldots \ (\text{xxx}).$$

Hence, if

$$\frac{d^2v}{dt^2} = \frac{B^2}{2v^n} \quad \ldots\ldots\ldots\ldots\ldots \quad (43)_n,$$

$$\frac{d^2T}{dt^2} = -\frac{C}{v^n} \quad \ldots\ldots\ldots\ldots\ldots \quad (44)_n,$$

then (xxx) divided by $\frac{1}{v^n}$ will become

$$\frac{B^2}{2}\left(\frac{x+T}{v}\right) + C + \frac{nA^2f^{n-2}}{v^{-1}}\frac{df}{dx} = 0 \ldots\ldots (\text{xxxi}).$$

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But $$f = f\left(\frac{x+T}{v}\right) = f(\xi);$$

therefore $$\frac{df}{dx} = \frac{df}{d\xi}\frac{d\xi}{dx} = v^{-1}\frac{df}{d\xi},$$

consequently (xxxi) may be written

$$\frac{B^2}{2}\xi + C + nA^2\{f(\xi)\}^{n-2}\frac{df}{d\xi} = 0 \ldots\ldots (59)_2,$$

which when $n=1$ agrees with (59). Multiplying into $d\xi$ and integrating, we find

$$\frac{B^2\xi^2}{4} + C\xi + \frac{nA^2}{n-1}\{f(\xi)\}^{n-1} = N \ldots\ldots (60)_2,$$

where N, like B and C, is an arbitrary constant. The formula fails when $n=1$, but this case has already been fully discussed. In virtue of (xxiv) equation $(60)_2$ becomes

$$\frac{B^2\xi^2}{4} + C\xi + \frac{nA^2}{n-1}(v\rho)^{n-1} = N \ldots\ldots (60)',$$

and $$P = \frac{1}{v}\left\{\frac{n-1}{nA^2}\left(N - C\xi - \frac{B^2\xi^2}{4}\right)\right\}^{\frac{1}{n-1}} \ldots\ldots (61)'.$$

108. In proceeding to the integration of $(43)_n$ and $(44)_n$, we may adopt at once the equations which, in Art. 6, are marked (45) to (53), both numbers inclusive. But the value of t therein contained must be altered in conformity with the article next following (*i.e.* Art. 109).

109. Multiplying either side of $(43)_n$ into $2\frac{dv}{dt}$, we have, on integration,

$$\left(\frac{dv}{dt}\right)^2 = \frac{B^2v^{1-n}}{1-n} + m = \frac{B^2v^{1-n} + m(1-n)}{1-n},$$

$$\frac{dt}{dv} = \left\{\frac{1-n}{B^2v^{1-n} + m(1-n)}\right\}^{\frac{1}{2}} \ldots\ldots (57)',$$

$$t = \int\left\{\frac{1-n}{B^2v^{1-n} + m(1-n)}\right\}^{\frac{1}{2}} dv \ldots\ldots (58)',$$

and in order to complete the solution, we must take steps analogous to those taken in Art. 9.

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A DEMONSTRATION OF A PROPERTY OF ELLIPTIC FUNCTIONS.

By WILLIAM WALTON, M.A., Fellow of Trinity Hall.

IF θ and ϕ are connected by the equation

$$F(c, \theta) + F(c, \phi) = F(c, \mu) \quad \ldots\ldots\ldots\ldots (1),$$

where μ is a constant, then will

$$\cos\theta \cos\phi - \sin\theta \sin\phi (1 - c^2 \sin^2\mu)^{\frac{1}{2}} = \cos\mu.$$

The differential equation

$$\frac{d\theta}{\sqrt{(1 - c^2 \sin^2\theta)}} + \frac{d\phi}{\sqrt{(1 - c^2 \sin^2\phi)}} = 0,$$

on the integration of which is based the demonstration of this well known property, will be seen by the following method of proof to belong intrinsically to Clairaut's form of differential equations.

The differential equation, cleared of radicals, is

$$d\theta^2 - d\phi^2 = c^2 (\sin^2\phi d\theta^2 - \sin^2\theta d\phi^2).$$

Multiplying by $\cos^2\theta - \cos^2\phi$, we have

$$(d\theta^2 - d\phi^2)(\cos^2\theta - \cos^2\phi) = c^2 (\cos^2\theta - \cos^2\phi)(\sin^2\phi d\theta^2 - \sin^2\theta d\phi^2).$$

Now the first member of the equation is equal to

$$\begin{aligned} &\sin(\phi + \theta).(d\theta + d\phi).\sin(\phi - \theta).(d\theta - d\phi) \\ &= - d\cos(\phi + \theta).d\cos(\phi - \theta) \\ &= dy^2 - dx^2, \end{aligned}$$

where x represents $\cos\theta \cos\phi$ and y represents $\sin\theta \sin\phi$.

Again, the second member of the equation is equal to c^2 multiplied by the product of the two expressions

$$(\cos\theta + \cos\phi)(\sin\phi d\theta + \sin\theta d\phi),$$

and

$$(\cos\theta - \cos\phi)(\sin\phi d\theta - \sin\theta d\phi).$$

The former of these expressions is equal to

$$\begin{aligned} &d(\sin\theta \sin\phi) + \sin\phi \cos\phi d\theta + \sin\theta \cos\theta d\phi \\ &= d(\sin\theta \sin\phi) + \sin\phi \cos\phi (\sin^2\theta + \cos^2\theta) d\theta \\ &\qquad + \sin\theta \cos\theta (\sin^2\phi + \cos^2\phi) d\phi \\ &= dy + xdy - ydx: \end{aligned}$$

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and the latter, as is evident by writing $\pi - \phi$ instead of ϕ, is evidently equal to

$$dy - xdy + ydx.$$

Thus the differential equation in θ and ϕ is transformed into

$$dx^2 - dy^2 = c^2\{(xdy - ydx)^2 - dy^2\},$$

whence $$y = xp - \frac{1}{c}\{(c^2 - 1)\, p^2 + 1\}^{\frac{1}{2}},$$

a differential equation of Clairaut's form.

Its integral, γ being a constant, is evidently

$$y = \gamma x - \frac{1}{c}\{(c^2 - 1)\, \gamma^2 + 1\}^{\frac{1}{2}},$$

or $$\sin\theta \sin\phi = \gamma \cos\theta \cos\phi - \frac{1}{c}\{(c^2 - 1)\, \gamma^2 + 1\}^{\frac{1}{2}}.$$

Now $\theta = \mu$ when $\phi = 0$: hence

$$\gamma \cos\mu = \frac{1}{c}\{(c^2 - 1)\, \gamma^2 + 1\}^{\frac{1}{2}},$$

whence $$\gamma^2 (1 - c^2 \sin^2\mu) = 1.$$

Hence the integral becomes

$$\cos\theta \cos\phi - (1 - c^2 \sin^2\mu)^{\frac{1}{2}} \sin\theta \sin\phi = \cos\mu.$$

September 16, 1870.

A DEMONSTRATION THAT EVERY EQUATION HAS A ROOT.

By William Walton, M.A., Fellow of Trinity Hall.

Let fx represent the polynomial of the theory of equations, and suppose the coefficients of the various powers of x to be possible quantities. Write $u + v\sqrt{(-1)}$ for x, u and v being possible quantities, and let

$$f\{u + v\sqrt{(-1)}\} = P + Q\sqrt{(-1)},$$

P and Q being possible quantities.

Differentiate this equation $\lambda + \mu$ times with regard to u: then, $\underset{\lambda+\mu}{f}(x)$ denoting the $(\lambda+\mu)^{\text{th}}$ derivative of $f(x)$,

$$\underset{\lambda+\mu}{f}\{u + v\sqrt{(-1)}\} = \frac{d^{\lambda+\mu}P}{du^{\lambda+\mu}} + \sqrt{(-1)}\,\frac{d^{\lambda+\mu}Q}{du^{\lambda+\mu}}:$$

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next differentiate the equation, λ times with regard to u and μ times with regard to v: then

$$(-1)^{\frac{1}{2}\mu} f_{\lambda+\mu}\{u+v\sqrt{(-1)}\} = \frac{d^{\lambda+\mu}P}{du^\lambda dv^\mu} + \sqrt{(-1)}\,\frac{d^{\lambda+\mu}Q}{du^\lambda dv^\mu}:$$

from these two results we see that

$$(-1)^{\frac{1}{2}\mu}\left\{\frac{d^{\lambda+\mu}P}{du^{\lambda+\mu}} + \sqrt{(-1)}\,\frac{d^{\lambda+\mu}Q}{du^{\lambda+\mu}}\right\} = \frac{d^{\lambda+\mu}P}{du^\lambda dv^\mu} + \sqrt{(-1)}\,\frac{d^{\lambda+\mu}Q}{du^\lambda dv^\mu}.$$

From this relation we have at once the two systems,

$$(\mu \text{ even})\ldots \left.\begin{aligned} \frac{d^{\lambda+\mu}P}{du^\lambda dv^\mu} &= (-1)^{\frac{1}{2}\mu}\,\frac{d^{\lambda+\mu}P}{du^{\lambda+\mu}} \\ \frac{d^{\lambda+\mu}Q}{du^\lambda dv^\mu} &= (-1)^{\frac{1}{2}\mu}\,\frac{d^{\lambda+\mu}Q}{du^{\lambda+\mu}} \end{aligned}\right\} \ldots\ldots\ldots\ldots (1);$$

and $$(\mu \text{ odd})\ldots \left.\begin{aligned} \frac{d^{\lambda+\mu}P}{du^\lambda dv^\mu} &= (-1)^{\frac{1}{2}(\mu+1)}\,\frac{d^{\lambda+\mu}Q}{du^{\lambda+\mu}} \\ \frac{d^{\lambda+\mu}Q}{du^\lambda dv^\mu} &= (-1)^{\frac{1}{2}(\mu-1)}\,\frac{d^{\lambda+\mu}P}{du^{\lambda+\mu}} \end{aligned}\right\} \ldots\ldots\ldots (2).$$

Let F represent P^2+Q^2. We proceed to ascertain the nature of what must certainly exist, a minimum value of F. Write $u+h$, $v+k$, respectively, for u, v; and let F' be the corresponding value of F. Then

$$F'-F = \left(h\frac{d}{du}+k\frac{d}{dv}\right)(P^2+Q^2) + \frac{1}{1.2}\left(h\frac{d}{du}+k\frac{d}{dv}\right)^2(P^2+Q^2)+\ldots.$$

This expression shews that the existence of a minimum value of F requires that

$$\left.\begin{aligned} P\frac{dP}{du} + Q\frac{dQ}{du} &= 0 \\ \text{and} \qquad P\frac{dP}{dv} + Q\frac{dQ}{dv} &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (3).$$

But, putting in (2), $\lambda=0$ and $\mu=1$, we see that $\frac{dP}{dv} = -\frac{dQ}{du}$ and $\frac{dQ}{dv} = \frac{dP}{du}$: hence, by (3), we must have, simultaneously, the systems

$$F.\frac{dP}{du}=0,\ F.\frac{dP}{dv}=0,\ F.\frac{dQ}{du}=0,\ F.\frac{dQ}{dv}=0.$$

Consequently either P and Q must be simultaneously zero, or

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$\frac{dP}{du}, \frac{dP}{dv}, \frac{dQ}{du}, \frac{dQ}{dv}$, must be simultaneously zero. Suppose, in the first place, that P and Q are not both zero; and choose the latter alternative. Suppose moreover in the expansion, by powers of h, k, for the value of $F' - F$, the lowest order of partial differential coefficients of P, Q, of which at least some are not zero, to be the $(2r)^{\text{th}}$. It is needless to consider the hypothesis of an odd order of differential coefficients not all vanishing; since, under these circumstances, there could not be any corresponding minimum value of F.

We have then

$$\lfloor 2r\,(F'-F) = \left(h\frac{d}{du}+k\frac{d}{dv}\right)^{2r}(P^2+Q^2)+\ldots\ \ldots(4).$$

Now, by the aid of Leibnitz' Theorem,

$$\begin{aligned}\frac{d^{2r}P^2}{du^{2r-\nu}dv^{\nu}} = \frac{d^{2r-\nu}}{du^{2r-\nu}}\Big(&P\frac{d^{\nu}P}{dv^{\nu}}+\frac{\nu}{1}\frac{dP}{dv}\frac{d^{\nu-1}P}{dv^{\nu-1}}\\ &+\frac{\nu(\nu-1)}{1.2}\frac{d^2P}{dv^2}\frac{d^{\nu-2}P}{dv^{\nu-2}}\\ &+\ldots\ldots\ldots\ldots\\ &+P\frac{d^{\nu}P}{dv^{\nu}}\Big)\\ = 2P\frac{d^{2r}P}{du^{2r-\nu}dv^{\nu}}&\end{aligned}$$

+ multiples of lower partial differential coefficients of P, which are all zero.

Hence, by (1),

$$\frac{d^{\nu}P^2}{du^{\nu-\nu}dv^{\nu}} = 2P(-1)^{\frac{1}{2}\nu}\frac{d^{2r}P}{du^{2r}},\ \ldots\ \nu \text{ even},\ \ldots\ldots\ldots(5),$$

and, by (2),

$$\frac{d^{2r}P^2}{du^{2r-\nu}dv^{\nu}} = 2P(-1)^{\frac{1}{2}(\nu+1)}\frac{d^{2r}Q}{du^{2r}},\ \ldots\ \nu \text{ odd},\ \ldots\ldots(6).$$

In precisely the same way, we have, paying attention to (1) and (2),

$$\begin{aligned}\frac{d^{2r}Q^2}{du^{2r-\nu}dv^{\nu}} &= 2Q\frac{d^{2r}Q}{du^{2r-\nu}dv^{\nu}}\\ &= 2Q(-1)^{\frac{1}{2}\nu}\frac{d^{2r}Q}{du^{2r}},\ \nu \text{ even},\ \ldots\ldots\ldots(7),\\ &= 2Q(-1)^{\frac{1}{2}(\nu-1)}\frac{d^{2r}P}{du^{2r}},\ \nu \text{ odd},\ \ldots\ldots(8).\end{aligned}$$

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By the relations (5), (6), (7), (8), the equation (4) becomes, C denoting the number of combinations of $2r$ things, taken ν at a time,

$$\begin{aligned}
\tfrac{1}{2}\lfloor\underline{2r}.(F'-F) = P\Big\{ & h^{2r}\frac{d^{2r}P}{du^{2r}} - C_1 h^{2r-1}k\ \frac{d^{2r}Q}{du^{2r}} \\
& - C_2 h^{2r-2}k^2\frac{d^{2r}P}{du^{2r}} \\
& + C_3 h^{2r-3}k^3\frac{d^{2r}Q}{du^{2r}} \\
& + C_4 h^{2r-4}k^4\frac{d^{2r}P}{du^{2r}} \\
& \cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots \\
& + (-1)^r C_1 hk^{2r-1}\frac{d^{2r}Q}{du^{2r}} \\
& + (-1)^r k^{2r}\frac{d^{2r}P}{du^{2r}}\Big\} \\
+ Q\Big\{ & h^{2r}\frac{d^{2r}Q}{du^{2r}} + C_1 h^{2r-1}k\ \frac{d^{2r}P}{du^{2r}} \\
& - C_2 h^{2r-2}k^2\frac{d^{2r}Q}{du^{2r}} \\
& - C_3 h^{2r-3}k^3\frac{d^{2r}P}{du^{2r}} \\
& + C_4 h^{2r-4}k^4\frac{d^{2r}Q}{du^{2r}} \\
& \cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots \\
& + (-1)^{r-1} C_1 hk^{2r-1}\frac{d^{2r}P}{du^{2r}} \\
& + (-1)^r k^{2r}\frac{d^{2r}Q}{du^{2r}}\Big\},
\end{aligned}$$

and therefore $\lfloor\underline{2r}(F'-F)$ is equal to

$$\left(P\frac{d^{2r}P}{du^{2r}} + Q\frac{d^{2r}Q}{du^{2r}}\right).[\{h+k\sqrt{(-1)}\}^{2r} + \{h-k\sqrt{(-1)}\}^{2r}]$$

$$+\frac{1}{\sqrt{(-1)}}\left(Q\frac{d^{2r}P}{du^{2r}} - P\frac{d^{2r}Q}{du^{2r}}\right).[\{h+k\sqrt{(-1)}\}^{2r} - \{h-k\sqrt{(-1)}\}^{2r}] + \ldots.$$

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Putting $$h = \rho \cos\theta, \quad k = \rho \sin\theta,$$

$$P\frac{d^{2r}P}{du^{2r}} + Q\frac{d^{2r}Q}{du^{2r}} = c \sin\alpha,$$

$$Q\frac{d^{2r}P}{du^{2r}} - P\frac{d^{2r}Q}{du^{2r}} = c \cos\alpha,$$

we have $\frac{1}{2}\lfloor 2r\,(F' - F) = c\rho^{2r} \sin(\alpha + 2r\theta)$ + terms of higher powers in ρ than the $(2r)^{\text{th}}$.

But, by properly varying the value of the arbitrary quantity θ, we may give negative values to the term $c\rho^{2r} \sin(\alpha + 2r\theta)$. Thus there is no minimum value of F under the present hypothesis. It follows therefore that, the existence of a minimum being obvious *à priori*, P and Q are capable of being zero simultaneously. This conclusion establishes the proposition that every equation has a root.

I have assumed in the above investigation that the coefficients of the various powers of x in $f(x)$ are possible. This restriction may readily be removed. In fact $f(x)$ may be represented by $\phi(x) + \sqrt{(-1)}\,\psi(x)$, where $\phi(x)$ and $\psi(x)$ involve only possible coefficients. Assume that $f(x) = 0$: then

$$(\phi x)^2 = -(\psi x)^2,$$

or $$(\phi x)^2 + (\psi x)^2 = 0.$$

This equation, by the proposition we have established, can always have a root $u + v\sqrt{(-1)}$. Hence either

$$\phi x + \sqrt{(-1)}\,\psi(x) = 0,$$

or $$\phi x - \sqrt{(-1)}\,\psi(x) = 0,$$

must have a root. Let the former have a root: then, putting $u + v\sqrt{(-1)}$ in this equation, we have a result of the form

$$P + Q\sqrt{(-1)} + \sqrt{(-1)}\,\{P' + Q'\sqrt{(-1)}\} = 0,$$

or $$P - Q' + \sqrt{(-1)}\,(P' + Q) = 0,$$

and therefore $$P = Q', \quad P' = -Q.$$

Substitute $u - v\sqrt{(-1)}$ in the latter of the two equations, and we have

$$P - Q\sqrt{(-1)} - \sqrt{(-1)}\,\{P' - Q'\sqrt{(-1)}\} = 0,$$

$$P - Q' - \sqrt{(-1)}\,(P' + Q) = 0,$$

a true result in virtue of the relations $P = Q'$, $P' = -Q$. Thus $f(x) = 0$ has a root, whether its coefficients are all possible or not.

September 16, 1869.

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ON EVOLUTES AND PARALLEL CURVES.

By Prof. CAYLEY.

IN abstract geometry we have a conic called the Absolute; lines which are harmonics of each other in regard to the absolute, or, what is the same thing, which are such that each contains the pole of the other in regard to the absolute, are said to be at right angles. Similarly, points which are harmonics of each other in regard to the absolute; or, what is the same thing, which are such that each lies in the polar of the other are said to be quadrantal.

A conic having double contact with the absolute is said to be a circle; the intersection of the two common tangents is the centre of the circle; the line joining the two points of contact, or chord of contact, is the axis of the circle.

Taking as a definition of equidistance that the points of a circle are equidistant from the centre, we arrive at the notion of distance generally, and we can thence pass down to that of equal circles; but the notion of equal circles may be established descriptively in a more simple manner:

Any two circles have an axis of symmetry, viz. this is the line joining their centres; and they have a centre of homology, viz. this is the intersection of their axes. They intersect in four points, lying in pairs on two lines through the centre of homology: they have also four tangents meeting in pairs in two points on the axis of symmetry. Now if the two lines through the centre of homology are harmonically related to the two axes; or, what is the same thing, if the two points on the axis of symmetry are harmonically related to the two centres, then the circles are *equal*.

Circles which are equal to the same circle are equal to each other, and the entire series of circles which are equal to a given circle, are said to be a system of circles of constant magnitude.

Starting from these general considerations, I pass to the question of evolutes and parallel curves: it will be understood that everything—lines at right angles, circles, poles, polars, reciprocal curves, &c.—refers to the absolute.

At any point of a curve we have a normal, viz. this is a line at right angles to the tangent; or, what is the same thing, it is the line joining the point with the pole of the

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tangent. The locus of the pole of the tangent is the reciprocal curve, and for any point of a given curve, the pole of the tangent at that point is the corresponding point of the reciprocal curve. Hence, also the normal is the line from the point to the corresponding point of the reciprocal curve. And the curve and its reciprocal have at corresponding points the same normal.

The envelope of the normals is the evolute; any curve having with the given curve the same normals (and therefore the same evolute) is a parallel curve; in other words, the parallel curve is any orthogonal trajectory of the normals of the given curve.

The parallel curve is also the envelope of a circle of constant radius having its centre on the given curve; or, again, it is the envelope of a circle of constant radius touching the given curve.

The theory in the above form is directly applicable to spherical or rather conical geometry; but in ordinary plane geometry the absolute degenerates into a point-pair, the two circular points at infinity, or say the points I, J; and this is a case that requires to be separately treated. The theory in the general case, the absolute a conic, is the more symmetrical and elegant, and it might appear advantageous to commence with this; but upon the whole I prefer the opposite course, and will commence with the case of plane geometry, the absolute a point-pair.

The subject connects itself with that of foci: I call to mind that a common tangent of the curve and the absolute is a focal tangent, and the intersection of two focal tangents a focus. In the case where the absolute is a point-pair, the focal tangents are the tangents from I to the curve, and the tangents from J to the curve, or say these are the I-tangents and the J-tangents; a focus is the intersection of an I-tangent and a J-tangent; the line IJ when it touches the curve, and (when the curve passes through I and J or either of them), the tangents at I or J to the curve are usually not reckoned as focal tangents; and other singular tangents, for instance, a double or stationary tangent through I or J is also excluded from the focal tangents; and the number of foci is of course reckoned accordingly, viz. it is the product of the number of the I-tangents into that of the J-tangents. So when the absolute is a conic; if this is touched by the curve, the common tangent at the point of contact is not reckoned as a focal tangent; and we may also exclude any singular tangents which touch the absolute;

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and the number of foci is reckoned accordingly, viz. it is equal to the number of pairs of focal tangents.

Let the Plückerian numbers for the given curve be $(m, n, \delta, \kappa, \tau, \iota)$, viz. m the order, n the class, δ the number of nodes, κ of cusps, τ of bitangents, ι of inflexions; and suppose moreover that D is the deficiency, and α the statitude; viz.

$$\alpha = 3m + \iota, \quad = 3n + \kappa;$$

$$2D = (m-1)(m-2) - 2\delta - 2\kappa, \quad = (n-1)(n-2) - 2\tau - 2\iota,$$

$$= n - 2m + 2 + \kappa, \quad = m - 2n + 2 + \iota,$$

$$= -2m - 2n + 2 + \alpha.$$

And let the corresponding numbers for the evolute be

$$(m'', n'', \delta'', \kappa'', \tau'', \iota''; D'', \alpha'').$$

These are most readily obtained, as in Clebsch's paper, "Ueber die Singularitäten algebraischer Curven," *Crelle*, t. LXIV. (1864), pp. 98–100, viz. it being assumed that we have

$$n'' = m + n, \quad \iota'' = 0,$$

then by reason that the evolute has a (1, 1) correspondence with the original curve, the two curves have the same deficiency, or writing this relation under the form

$$m'' - 2n'' + \iota'' = m - 2n + \iota,$$

we have $m'' = 3m + \iota, \ = \alpha$; and the Plückerian relations then give the values of κ'', δ'', τ''.

In regard to these equations $n'' = m + n$, $\iota'' = 0$, I remark that if we have two curves of the orders m, m', and on these points P, Q having an (α, α') correspondence, the line PQ envelopes a curve of the class $m\alpha' + m'\alpha$, and the number of inflexions is in general $= 0$. Now in the present case, taking P on the given curve and Q the point of intersection with IJ of the normal (or harmonic of the tangent), the orders of the curves are $(m, 1)$, and the correspondence is $(n, 1)$; whence as stated $m'' = m + n$, $\iota'' = 0$.

The formulæ thus are

$$m'' = \alpha, \qquad \alpha'' = 3\alpha,$$

$$n'' = m + n, \qquad D' = D,$$

$$\iota'' = 0,$$

$$\kappa'' = -3m - 3n + 3\alpha,$$

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in which formulæ it is assumed that the curve has no special relations to the points I, J; or, what is the same thing, that the line IJ intersects the curve in m points distinct from each other, and from the points I, J.

It is to be added (see Salmon's *Higher Plane Curves*, 1852, pp. 109 et seq.) that m of the κ'' cusps arise from the intersections of the curve with IJ, these cusps being situate on the line IJ, and each of them the harmonic of one of the intersections in question, and the cuspidal tangent being for each of them the line IJ. The intersections of the evolute by the line IJ are these m cusps each 3 times, and besides ι points arising from the ι inflexions of the curve; viz. at any inflexion the two consecutive normals intersect in a point on the line IJ, being in fact the harmonic of the intersection of IJ with the tangent at the inflexion. It was in this manner that Salmon obtained the number $3m + \iota$ of the points at infinity of the evolute, that is the expression $m'' = 3m + \iota\,(= \alpha)$ for the order of the evolute.

The remaining $-4m - 3n + 3\alpha$ cusps arise from the points on the curve where there is a circle of 4-pointic intersection, or contact of the third order, and in this manner the number of them was found, Salmon, p. 113, in the particular case of a curve without nodes or cusps, and generally in Zeuthen's *Nyt Bidrag*, &c., p. 91; the number of the points in question, in the foregoing form $-4m - 3n + 3\alpha$, is also obtained in my Memoir, "On the curves which satisfy given conditions," *Phil. Trans.*, 1868, pp. 75–143, see p. 97.

It is further to be noticed that the $m + n$ tangents to the evolute from either of the points I, J are made up of the line IJ counting m times (in respect that it is a tangent at each of the above mentioned m cusps) and of the n tangents from the points in question to the original curve. Or taking the two points I, J conjointly, say the $2m + 2n$ common tangents of the absolute and the evolute are made up of the line IJ (or axis of the absolute) counting m times, and of the $2n$ focal tangents of the original curve. The focal tangents of the original curve and of the evolute are thus the same $2n$ lines; and the two curves have the same foci.

The above are the ordinary values of $m'', n'', \iota'', \kappa''$, but if the given curve touch the line IJ, then the evolute has at the point of contact an inflexion, the stationary tangent being the line IJ; and if the given curve pass through one or other of the points I, J, the evolute has in this case an inflexion on the tangent at the point in question, this tangent being the stationary tangent of the evolute: but ob-

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serve that the inflexion is not at the point I or J in question: and for each inflexion there is a diminution = 1 in the class, 3 in the order, and 5 in the number of cusps. Suppose that the point I is a f_1-tuple point on the given curve; then the evolute has f_1 inflexions; and similarly if the point J is a f_2-tuple point on the given curve, then the evolute has f_2 inflexions. Hence writing $f_1+f_2=f$, we have thus f inflexions; and if moreover the number of contacts with the line IJ be $=g$, then we have on this account g inflexions; or in all $f+g$ inflexions, and the formulæ become

$$\begin{aligned} m'' &= \alpha - 3f - 3g, \\ n'' &= m+n - f - g, \\ \iota'' &= f + g, \\ \kappa'' &= -3m - 3n + 3\alpha - 5f - 5g. \end{aligned}$$

It is to be noticed here that the number of the intersections of the given curve with the line IJ (other than the points I, J and the points of contact) is $= m - f_1 - f_2 - 2g$, that is $m - f - 2g$: each of these gives as before a cusp on the evolute, the cuspidal tangent being IJ; we have besides on the line IJ (in respect of the g contacts) g inflexions, the stationary tangent being the line IJ; and each of the ι inflexions gives for the evolute a point on the line IJ; hence the whole number of intersections with the line IJ is $3(m-f-2g)+3g+\iota$, $=3m+\iota-3f-3g$, which is thus the order of the evolute.

The tangents from the point I or J to the evolute are the line IJ counting $m-f-2g$ times in respect of the cusps on this line and $2g$ times in respect to the inflexions, that is $m-f$ times; the tangents at the point in question to the given curve each twice as touching the evolute at an inflexion, $2f_1$ or $2f_2$: and the remaining $n-2f_1-g$, or $n-2f_2-g$ tangents from the point in question to the given curve; the whole number is thus $(m-f)+2f_1+(n-2f_1-g)$ or $(m-f)+2f_2+(n-2f_2-g)$, $=m+n-f-g$, the class of the evolute. The two values of n'' give

$$2n'' = 2m + (n-2f_1-g) + (n-2f_2-g),$$

viz. twice the class of the evolute = twice the order of the curve + the number of the focal I-tangents + that of the focal J-tangents; but this is not true for all relations whatever of the curve to the absolute.

The tangents from I to the given curve (excluding the line IJ and the tangents at I) are $n-2f_1-g$ tangents; and similarly the tangents from I to the evolute (excluding the

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line IJ and the stationary tangents through I) are the same $n-2f_1-g$ tangents; say the curve and the evolute have the same $n-2f_1-g$ I-tangents. Similarly they have the same $n-2f_2-g$ J-tangents; or together the same $2(n-f_1-f_2-g)$, $=2(n-f-g)$ focal tangents. And the curve and evolute have the same $(n-2f_1-g)(n-2f_2-g)$ foci.

The foregoing specialities f and g refer, g to the ordinary contacts of the line IJ with the curve, viz. the curve is supposed to have with the curve at an ordinary or non-singular point thereof a contact or 2-pointic intersection, and f, that is f_1 or f_2, to the multiple points having f_1 or f_2 ordinary branches, none of them touching the line IJ. Thus the formulæ do not apply to the cases of IJ passing through a node or a cusp of the given curve, or touching it at an inflexion; nor to the cases where at I or J the curve touches IJ, or where there is at I or J an ordinary double point with one of its branches touching IJ, or where there is at I or J a cusp, where the cuspidal tangent is or is not IJ.

It is easy to see that in the case of a multiple point of any kind whether situate on IJ or at I or J, each branch of the curve produces its own separate effect on the singularities of the evolute: thus if we have on IJ a double point neither branch touching IJ, then the separate effect of each branch is *nil*, therefore the effect of the double point is also *nil:* but if one branch touch the line IJ, then the whole effect is the same as if we had this branch only; viz. we have here the case $g=1$. And so if there is at I or J a double point with one branch touching the line IJ, then the effect of this branch is as if we had this branch only (a case not yet investigated) but the other branch is the case $f=1$. And so if we have at I or J a double point with two ordinary branches touching each other (tacnode or, if the two branches have a contact higher than the first order, oscnode), then if the branches do not touch the line IJ the case is $f=2$, but if they do, then the effect is twice that of an ordinary branch touching IJ. In support of these conclusions, observe that such multiple points, with ordinary branches, present themselves in the case of two or more curves which intersect or touch each other in any manner; and that the evolute of a system of two or more curves is simply the system of the evolutes of the several curves.

It follows as regards the relations of the given curve to the points I, J, and the effect thereby produced on the evolute, we only need to consider the case of a single branch; viz. the cases are

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the given curve intersects the line IJ at a point other than I or J, and belonging thereto there is a branch ordinary or singular,

not touching IJ,

touching IJ;

and the given curve passes through the point I or J, and belonging thereto there is a branch ordinary or singular,

not touching IJ,

touching IJ.

I have succeeded in determining the effect, not for a singular branch of any kind whatever, but for branches of the form $y = x^k$, $y^{k-1} = x^k$; viz. $k = 2$ each of these is an ordinary branch, $k = 3$, the first $y = x^3$ is an inflexional branch and the second $y = x^{\frac{3}{2}}$ a cuspidal branch; and so $k > 3$ the two branches are respectively inflexional and cuspidal of a higher order. I do this very simply by consideration of the curve $x^{k-1}z = y^k$.

The curve in question $x^{k-1}z = y$, is a unicursal curve, and it has a reciprocal of the same form $X^{k-1}Z = Y^k$, hence

$$m = n = k; \quad 0 = n - 2m + 2 + \kappa,$$

whence $\quad \iota = \kappa = k - 2, \ \tau = \delta = \frac{1}{2}(k-2)(k-3);$

viz. the point $x = 0$, $y = 0$ is a cusp equivalent to $k-2$ cusps and $\frac{1}{2}(k-2)(k-3)$ nodes; and the point $z = 0$, $y = 0$ is an inflexion equivalent to $k-2$ inflexions and $\frac{1}{2}(k-2)(k-3)$ bitangents.

The equation $U = x^{k-1}z - y^k = 0$, of the curve is satisfied by writing therein $x : y : z = 1 : \theta : \theta^k$; and these values give

$$d_x U : d_y U : d_z U = (k-1)x^{k-2}z : -ky^{k-1} : x^{k-1}$$

$$= (k-1)\theta^k : -k\theta^{k-1} : 1.$$

Taking the coordinates of I, J to be (α, β, γ) and $(\alpha', \beta', \gamma')$ respectively, and X, Y, Z as current coordinates, the equation of the normal at the point (x, y, z) of the curve $U = 0$ is readily found to be

$$(\alpha' d_x U + \beta' d_y U + \gamma' d_z U) \begin{vmatrix} X, & Y, & Z \\ \alpha, & \beta, & \gamma \\ x, & y, & z \end{vmatrix}$$

$$+ (\alpha d_x U + \beta d_y U + \gamma d_z U) \begin{vmatrix} X, & Y, & Z \\ \alpha', & \beta', & \gamma' \\ x, & y, & z \end{vmatrix} = 0.$$

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Hence for the curve in question, the equation of the normal is

$$\{(k-1)\alpha'\theta^k - k\beta'\theta^{k-1} + \gamma'\}\{(\beta X - \alpha Y)\theta^k + (\alpha Z - \gamma X)\theta + (\gamma Y - \beta Z)\}$$
$$+ \{(k-1)\alpha\theta^k - k\beta\theta^{k-1} + \gamma\}$$
$$\times \{(\beta' X - \alpha' Y)\theta^k + (\alpha' Z - \gamma' X)\theta + (\gamma' Y - \beta' Z)\} = 0,$$

or expanding and reducing, this equation is

$$\theta^{2k}.(k-1)\ \{(\alpha\beta' + \alpha'\beta)X - 2\alpha\alpha' Y\}$$
$$+ \theta^{2k-1}.\ -k\{2\beta\beta' X - (\alpha\beta' + \alpha'\beta)Y\}$$
$$+ \theta^{k+1}.(k-1)\{2\alpha\alpha' Z - (\alpha\gamma' + \alpha'\gamma)X\}$$
$$+ \theta^k.\{(k+1)(\beta\gamma' + \beta'\gamma)X + (k-2)(\gamma\alpha' + \gamma'\alpha)Y - (2k-1)(\alpha\beta' + \alpha'\beta)Z\}$$
$$+ \theta^{k-1}.\ -k\{(\beta\gamma' + \beta'\gamma)Y - 2\beta\beta' Z\}$$
$$+ \theta\ .\ \{(\gamma\alpha' + \gamma'\alpha)Z - 2\gamma\gamma' X\}$$
$$+ \{2\gamma\gamma' Y - (\beta\gamma' + \beta'\gamma)Z\} = 0,$$

where k is a positive integer not less than 2; hence except in the case $k=2$, all the terms θ^{2k}, θ^{2k-1}, ... θ, θ^0, have different indices, and the coefficients $k-1$, k, &c. none of them vanish; if however $k=2$, then the terms θ^{2k-1}, θ^{k+1} coalesce into a single term, as do also the terms θ^{k-1} and θ; moreover the coefficient $k-2$ is $=0$.

The evolute is the envelope of the line represented by the foregoing equation, considering therein θ as an arbitrary parameter; viz. the equation is obtained by equating to zero the discriminant of the foregoing equation in θ. Hence in general the class of the evolute is $= 2k$, and its order is $= 2(2k-1)$; results which agree with the formulæ for n'', m'', since in the present case $m+n$, $= k+k$, $= 2k$, $\alpha = 3n+\kappa$, $= 3k+(k-2)$, $= 4k-2$. And moreover there are not any inflexions, $\iota'' = 0$ as before.

The equation may however contain a factor in θ independent of (X, Y, Z), and throwing out this factor, say its order is $= s$, the expression for the class is $2k-s$, $= m+n-s$, and that for the order is $4k-2-2s$, $= \alpha - 2s$. Moreover, in the original equation or in the equation thus reduced, it may happen that the equation will on writing therein $\Omega = 0$ (Ω a linear function of X, Y, Z) acquire a factor of the order ω, independent of (X, Y, Z); the line $\Omega = 0$ is in this case a stationary tangent $= \omega - 1$ inflexions; and the discriminant

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contains the factor $\Omega^{\omega-1}$, which may be thrown out; that is we have here $n''=2k-s$, $\iota''=\omega-1$, $m''=4k-2-2s-(\omega-1)$; agreeing with the relation $m''-2n''+2+\iota''=0$ which holds good in virtue of the evolute being a unicursal curve. It is in this manner that the values of m'', n'', ι'' are obtained in the several cases to be considered, viz.:

A_k Inflexion situate on IJ, which is not a tangent.
B_k Inflexion situate on IJ, which is a tangent.
C_k Cusp situate on IJ, which is not a tangent.
D_k Cusp situate on IJ, which is a tangent.
P_k Inflexion at J, IJ not a tangent.
Q_k Inflexion at J, IJ a tangent.
R_k Cusp at J, IJ not a tangent.
S_k Cusp at J, IJ a tangent.

The results being respectively as follows:

	A_k	B_k	C_k	D_k	P_k	Q_k	R_k	S_k
$m''=\alpha-$	0	$3k-3$	$k-2$	$k+1$	k	$3k-2$	$2k-2$	$2k$
$n''=m+n-$	0	$k-1$	0	1	1	k	$k-1$	k
$\iota''=0+$	0	$k-1$	$k-2$	$k-1$	$k-2$	$k-2$	0	0
$\kappa''=-3m-3n+3\alpha-$	0	$5k-5$	$2k-4$	$2k+1$	$2k-1$	$5k-4$	$3k-3$	$3k$

	A_2C_2	B_2D_2	A_3	B_3	C_3	D_3	$\overline{P}_2\overline{R}_2$	Q_2S_2	P_3	Q_3	R_3	S_3
$m''=\alpha-$	0	3	0	6	0	4	3	4	3	7	4	6
$n''=m+n-$	0	1	0	2	0	1	1	2	1	3	2	3
$\iota''=0+$	0	1	0	2	0	2	1	0	1	1	0	0
$\kappa''=-3m-3n+3\alpha-$	0	5	0	10	0	7	5	6	5	11	6	9

read for instance in B_k, $m''=\alpha-(3k-3)$, $n''=m+n-(k-1)$, $\iota''=0+(k-1)$, $\kappa''=-3m-3n+3\alpha+(5k-5)$; and so in other cases.

A_2C_2 (that is indifferently A_2 or C_2) is when there is on IJ an ordinary point, IJ not a tangent; and so B_2D_2 when there is on IJ an ordinary point, IJ a tangent. Similarly P_2R_2 when there is at J an ordinary point, IJ not a tangent; only instead thereof I have written $\overline{P}_2\overline{R}_2$ to indicate that (for a reason which will appear) the numbers are *not* deducible from those for P_2 or R_2 by writing therein $k=2$; and Q_2S_2 is when there is at J an ordinary point, IJ a tangent.

For the case A_k we have to take the line IJ passing through the inflexion; the condition for this is $\beta\gamma'-\beta'\gamma=0$: there is no speciality, or we have $n''=2k$, $m''=4k-2$, $\iota''=0$; whence also $\kappa''=0$; the value of κ'' being in every case deduced from those of m'', n'', ι'' by the formula

$$3m''+\iota''=3n''+\kappa''.$$

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Case B_k. I write $\gamma=\gamma'=0$, the equation of the normal is

$$\begin{array}{l|l}
\theta^{2k}\,(k-1)\,\{(\alpha\beta'+\alpha'\beta)\,X-2\alpha\alpha'\,Y) & \theta^{k+1}(X,\ Y)\\
+\,\theta^{2k-1}.-k\,\{2\beta\beta'X-(\alpha\beta'+\alpha'\beta)\,Y\} & +\ \theta^{k}\,(X,\ Y)\\
+\,\theta^{k+1}.\,(k-1)\,2\alpha\alpha'Z & +\ \theta^{2}\quad Z\\
+\,\theta^{k}.-(2k-1)\,(\alpha\beta'+\alpha'\beta)\,Z & +\ \theta\quad Z\\
+\,\theta^{k-1}\quad k.\quad 2\beta\beta'Z\qquad\qquad =0, & +\qquad Z=0,
\end{array}$$

where throwing out the factor θ^{k-1}, the form is as shown on the right hand. Writing $Z=0$, we have a factor θ^{k}, whence $\iota''=k-1$, and then $n''=k+1$, $m''=2k-(k-1)=k+1$, agreeing with the table. The process holds good for $k=2$.

Case C_k. I write $\beta=\beta'=0$; this brings as well the inflexion as the cusp upon the line IJ; but it has been seen (Case A_k) that there is not any reduction on account of this position of the inflexion, hence the whole effect will be due to the cusp. The equation is

$$\begin{array}{l|l}
\theta^{2k}\ (k-1)\,\{-2\alpha\alpha'\,Y\} & \theta^{2k}\quad Y\\
+\,\theta^{k+1}\,(k-1)\,\{2\alpha\alpha'Z-(\alpha\gamma'+\alpha'\gamma)\,X\} & +\,\theta^{k+1}\,(Z,\ X)\\
+\,\theta^{k}\ (k-2)\,(\gamma\alpha'+\gamma'\alpha)\,Y & +\,\theta^{k}\,(k-2)\,Y\\
+\,\theta\quad .\qquad \{(\gamma\alpha'+\gamma'\alpha)Z-2\gamma\gamma'X\} & +\,\theta\ \ (Z,\ X)\\
+\qquad 2\gamma\gamma'\,Y=0 & +\,Y=0,
\end{array}$$

so that here $n''=2k$. On writing $Y=0$, there is a factor $\left(1-\dfrac{\theta}{\infty}\right)^{k-1}$ thrown out (indicated by the reduction of the order from $2k$ to $k+1$), whence

$$\iota''=k-2,\quad m''=2\,(2k-1)-(k-2),\quad =3k.$$

The process holds good for $k=2$.

Case D^k. We may write $\alpha=\alpha'=0$; the equation is

$$\begin{array}{l|l}
\theta^{2k-1}.-k.2\beta\beta'X & \theta^{2k-1}X\\
+\,\theta^{k}\ .\ (k+1)\,(\beta\gamma'+\beta'\gamma)\,X & +\,\theta^{k}\ X\\
+\,\theta^{k-1}.-k[(\beta\gamma'+\beta'\gamma)\,Y-2\beta\beta'Z] & +\,\theta^{k-1}\,(Y,\ Z)\\
+\,\theta\quad .-2\gamma\gamma'X & +\,\theta\quad X\\
\qquad +2\gamma\gamma'\,Y-(\beta\gamma'+\beta'\gamma)\,Z=0, & +\qquad (Y,\ Z)=0,
\end{array}$$

so that $n''=2k-1$. Writing $X=0$, we have the factor $\left(1-\dfrac{\theta}{\infty}\right)^{k}$ (indicated by the reduction of order from $2k-1$ to

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$k-1$), whence $\iota''=k-1$, and then $m''=(4k-2)-2-(k-1)$, $=3k-3$, agreeing with the table. The process holds good for $k=2$.

Case P_k. We have $\beta'=\gamma'=0$; the equation is

$$\begin{array}{l|l}
\theta^{2k}\ (k-1)\{\alpha'\beta X-2\alpha\alpha' Y\} & \theta^{2k-1}(X,\ Y)\\
+\theta^{2k-1}.-k\ \{-\alpha'\beta Y\} & +\theta^{2k-2}\ Y\\
+\theta^{k+1}(k-1)\{-\alpha'\gamma X\} & +\theta^{k}\ \ X\\
+\theta^{k}\ (k-2)\,\gamma\alpha' Y-(2k-1)\,\alpha'\beta Z & +\theta^{k-1}(k-2)\ Y+Z\\
+\theta.\qquad\quad \gamma\alpha' Z=0 & +\qquad Z=0,
\end{array}$$

so that here $n''=2k-1$. Writing $Z=0$, we have the factor θ^{k-1}, whence $\iota''=k-2$, $m''=4k-4-(k-2)=3k-2$, agreeing with the table. If, however, $k=2$, then on writing $Z=0$ the equation (instead of the factor θ^{k-1}) acquires the factor $\theta^{k}\,(=\theta^2)$; so that here $n''=3$, $\iota''=1$, $m''=3$, agreeing with the column $\overline{P}_2\overline{R}_2$ of the table.

Case Q_k. We have $\gamma=0$, $\beta'=\gamma'=0$, viz. $\beta'=0$ in the formulæ of B_k. The equation is

$$\begin{array}{l|l}
\theta^{2k}\,(k-1)\,\alpha'\beta Y & \theta^{k}\ X\\
+\theta^{2k-1}.-k\,(-\alpha'\beta Y) & +\theta^{k-1}\,Y\\
+\theta^{k+1}\,(k-1)\,2\alpha\alpha' Z & +\theta\ \ Z\\
+\theta^{k}.-(2k-1)\ \alpha'\beta Z=0, & +\qquad Z=0,
\end{array}$$

so that $n''=k$. For $Z=0$ there is the factor θ^{k-1}, hence $\iota''=k-2$, $m''=2\,(k-1)-(k-2)$, $=k$. The process holds good for $k=2$.

Case R_k. We have $\beta=0$, $\alpha'=0$, $\beta'=0$, viz. $\alpha'=0$ in the formulæ of C_k. The equation is

$$\begin{array}{l|l}
\theta^{k+1}\,(k-1)\,(-\alpha\gamma' X) & \theta^{k+1}X\\
+\theta^{k}\ \ (k-2)\,(\alpha\gamma' Y) & +\theta^{k}\ Y(k-2)\\
+\theta.\qquad\quad (\alpha\gamma' Z-2\gamma\gamma' X) & +\theta\ \ (Z,\ X)\\
+\qquad\quad 2\gamma\gamma' Y=0, & +\qquad Y=0,
\end{array}$$

so that $n''=k+1$, $\iota''=0$, $m''=2k$. But observe that in the particular case $k=2$, the form is $\theta^3X+\theta\,(Z,\ X)+Y=0$, the term θ^2Y disappearing on account of the factor $k-2$. Here on writing $X=0$, there is a factor $\left(1-\dfrac{\theta}{\infty}\right)^2$ (indicated by the reduction of order from 3 to 1), hence $\iota''=1$, $n''=3$, $m''=2.2-1=3$, agreeing with the column $\overline{P}_2\overline{R}_2$.

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Case S_k. We have $\alpha=0$, $\alpha'=\beta'=0$, viz. $\beta'=0$ in the formulæ of D_k. The equation is

$$\begin{array}{llll} \theta^k & (k+1)\beta\gamma' X & & \theta^k\ X \\ +\theta^{k-1} & .-k\beta\gamma' Y & & +\theta^{k-1} Y \\ +\theta & .-2\gamma\gamma' X & & +\theta\ X \\ + & 2\gamma\gamma' Y-\beta\gamma' Z=0, & & +\ Y+Z=0, \end{array}$$

so that $n''=k$, $\iota''=0$, $m''=2k-2$. The process applies to the case $k=2$.

As to the formula for A_2, $B_2 \ldots S_2$, there is nothing special in these; they are simply deduced from those for A_k, $B_k \ldots S_k$ by writing therein $k=3$. And we have thus the foregoing series of formulæ, which will apply to the greater part of the cases which ordinarily arise. For instance suppose there is at I or J a triple point = cusp + 2 nodes; there is here an ordinary branch and a cuspidal (ordinary cuspidal) branch and according as IJ touches neither branch, the ordinary branch, or the cuspidal branch, the corrections to m'', n'', ι'', κ'' are $\overline{R}_2+R_2$, S_2+R_3, R_2+S_3 respectively. Observe moreover that A_2C_2 is no speciality, B_2D_2 is the speciality $g=1$, $\overline{P}_2\overline{R}_2$ the speciality $f=1$.

There is a remarkable case in which the fundamental assumption of the (1, 1) correspondence of the evolute with the original curve ceases to be correct. In fact, in the case about to be considered of a parallel curve; the parallel to any given curve is *in general* a curve not breaking up into two distinct curves of the same order with such given curve, and when this is so (viz. when the parallel curve does not break up) each normal of the parallel curve is a normal at two distinct points thereof: the evolute of the parallel curve is thus the evolute of the given curve taken twice; and the parallel curve and its evolute have not a (1, 1) but (1, 2) correspondence. Hence, (m, n, δ, κ, τ, ι) the unaccented letters referring to the parallel curve, or say rather to a curve which has a (1, 2) correspondence with its evolute, and, as before, the twice accented letters to the evolute, it is *not true* that $m''-2n''+\iota''=m-2n+\iota$; it will subsequently appear that the values of m'', n'' are correct, those of ι'', κ'' suffering a modification; viz. the formulæ are

$$\begin{array}{rlll} m''= & \alpha & -3f-3g, \\ n''= & m+n & -f-g, \\ \iota''= & & f+g-\Theta, \\ \kappa''= & -3m-3n+3\alpha & -5f-5g-\Theta, \end{array}$$

where, for the present, I leave Θ undetermined.

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Coming now to the parallel curve, let the numbers in regard to it be $m', n', \delta', \kappa', \tau', \iota'$; α', D'. Supposing in the first instance that the given curve does not stand in any special relation to I, J, the formulæ are

$$\begin{aligned} m' &= 2m + 2n, & \alpha &= 6n + 2\alpha, \\ n' &= 2n, & 2D' &= -4m + 2 + 2\alpha, \\ \iota' &= 2\alpha - 6m, & &= -2m + 2n + \alpha. \\ \kappa' &= 2\alpha. \end{aligned}$$

Considering the parallel curve as the envelope of a circle of constant radius having its centre on the given curve, it appears (*e.g.* by consideration of the case of the ellipse) that when the radius of the circle is $=0$, there is not any depression in the order of the parallel curve, but that the parallel curve reduces itself to the given curve twice, together with the system of tangents from the points I, J to the given curve: the order of the parallel curve is thus $m' = 2m + 2n$.

To find the class, consider the tangents from a given point to the parallel curve; about the point as centre describe a circle, radius k; then the tangents in question are respectively parallel to, and correspond each to each with, the common tangents of the circle and the given curve, and the number of these is $= 2n$, that is $n' = 2n$.

Each inflexion of the given curve gives rise to two inflexions of the parallel curve; and the inflexions of the parallel curve arise in this way only: that is $\iota' = 2\iota$, $= 2\alpha - 6m$. And the Plückerian relations then give $\kappa' = 2\alpha$; a value which may be investigated independently.

Attending now to the singularities f and g; the values of n', ι' are unaltered: to obtain m' we as before consider the particular case where the radius of the variable circle is $=0$: the parallel curve here breaks up into the original curve, together with the focal tangents from the points I, J; viz. we have

$$m' = 2n + (n - 2f_1 - g) + (n - 2f_2 - g), \quad = 2m + 2n - 2f - 2g;$$

and knowing m', n', ι' we have κ'.

The points on IJ of the original curve are I, J counting as f_1 and f_2 respectively; or together as f points: the points of contact counting as $2g$: and besides $m - f - 2g$ points. As regards the parallel curve, we have the same points on IJ; but here I is $(n-g)$ tuple point, having in respect of each branch of the f_1-tuple point on the original curve a pair

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of tangents meeting each other and in respect of each of the tangents it meets the given curve a single time, [illegible] and this counting i times similarly it counts $i-j$ times. Hence also for the parallel curve $j = f = \nu - \jmath$. In respect of each of the points j, we have a point where there are two tangents meeting each other and the line ∞, and thus counting 2 times as together as $2j$: moreover, in account of the two tangents at each of these points $j = 2j$. Lastly, each of the $n - f - 2j$ points x is a node on the parallel curve: and we have $\nu = 2\nu - j - 2j - 2 \quad n - f - 2j$. $- 2n + 2i - 2f - 2j$ as above.

Let us take also the formulæ

$$w = 2w + 2i - 2f - 2j,$$

$$w = 2w,$$

$$i = 3w + 2n, \quad = 2i,$$

$$\kappa = 2n + 6f - 6j, \quad = 6n - 2\kappa - 6f - 6j,$$

$$f = 2n - 2j,$$

$$\jmath = 2j,$$

where observe that m is $= 2n$; that is, twice the class of the evolute; which relation however is not true for a curve with any singularities whatever; and further that $n - f - \jmath$ is $= 0$.

The case of a curve for which $n - f - g = 0$ is very interesting and remarkable. Recurring to the formulæ for the evolute, we have here $m' = \kappa$, $n' = n$, $i' = n$, $\kappa' = n - 3n + 3\kappa$. And for the parallel curve $m = 2m$, $n' = 2n$, $i' = 2i$, $\kappa' = 2\kappa$, $f' = 2f$, $\jmath' = 2\jmath$; formulæ which lead to the assumption that the parallel curve here breaks up into two distinct curves, each such as the given curve.

Observe further that for a curve possessing the singularities f and $\jmath$, but where $n - f - g$ is not $= 0$; then for the parallel curve we have as above $n' - f' - g' = 0$; or the parallel of the parallel curve should, according to the assumption, break up into two distinct curves such as the parallel curve; this is of course correct.

Consider the evolute of the parallel curve: since for the parallel curve $n' - f' - \jmath' = 0$, the formulæ for the evolute thereof (viz. those containing the undetermined quantity Θ) are $m''' = \kappa'$, $n''' = m'$, $i''' = n' - \Theta$, $\kappa''' = n' - 3m' + 3\kappa' - \Theta$, or

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substituting for m', n', κ' their values, and comparing with the formulæ in regard to the evolute we have

$$\begin{aligned}
m''' &= 2\alpha - 6f - 6g &&= 2m'',\\
n''' &= 2m + 2n - 2f - 2g &&= 2n'',\\
\iota''' &= 2n - \Theta &&= 2\iota'' + 2(n-f-g) - \Theta,\\
\kappa''' &= -6m - 4n + 6\alpha - 12f - 12g - \Theta &&= 2\kappa'' + 2(n-f-g) - \Theta,
\end{aligned}$$

where m'', n'', ι'', κ'' refer to the evolute. Hence by assuming $\Theta = 2(n-f-g)$, the values of m''', n''', ι''', κ''' become $2m''$, $2n''$, $2\iota''$, $2\kappa''$, viz. the evolute of the parallel curve is the evolute of the original curve taken twice. Observe that in the foregoing value of Θ, the letters n, f, g refer not to the parallel curve, the evolute whereof is under consideration, but to the curve from which such parallel curve was derived; this value $\Theta = 2(n-f-g)$ is not a value of Θ applicable to be substituted in the evolute-formulæ for the case of a curve which has with its evolute a (1, 2) correspondence.

Instead of the foregoing case of the f, viz. f- and g-singularities we may, as regards the parallel curve, consider the original curve as having any I- and J-singularities whatever. Suppose in this case (excluding always the line IJ and the tangents at I or J) the number of tangents from I to the curve is $= n - I$, and the number of tangents from J to the curve is $= n - J$, then when the radius of the variable curve is $= 0$, the parallel curve becomes the original curve twice together with the $(n-I) + (n-J)$, $= 2n - I - J$ tangents; so that the order is $m' = 2m + 2n - I - J$*; we have, as before, $n' = 2n$ and $\iota' = 2\iota$, and these values give κ', so that the equations are

$$\begin{aligned}
m' &= 2m + 2n - I - J,\\
n' &= 2n,\\
\iota' &= -6m + 2\alpha, &&= 2\iota,\\
\kappa' &= 2\alpha - I - J, &&= 6n + 2\kappa - 3I - 3J.
\end{aligned}$$

Suppose $2n - I - J = 0$; this implies $n - I = 0$, $n - J = 0$ since neither $n - I$ nor $n - J$ can be negative; viz. that there are no I- or J-tangents; and conversely, when this is the case $2n - I - J = 0$: and we have then m', n', ι', $\kappa' = 2m$, $2n$, 2ι, 2κ; viz. it is assumed, as before, that the parallel curve breaks

* That the order of the evolute is not (in the case of a curve with any singularities whatever) one-half this, or $= m + n - \frac{1}{2}(I + J)$, is at once seen by remarking that there is no reason why $I + J$ should be even.

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up into two distinct curves such as the original curve; that is, the condition in order that the parallel curve should break up, is that the original curve has no focal tangents. Observe that the number of foci is $=(n-I)(n-J)$ which is $=0$ if only $n-I=0$ or $n-J=0$; but as regards real curves $I=J$, so that the equations $n-I=0$ and $n-J=0$ are one and the same equation, satisfied if $(n-I)(n-J)=0$; so that for a real curve without foci (real or imaginary) the parallel curve will break up. An instance given to me by Dr. Salmon is the curve $x^{\frac{1}{3}}+y^{\frac{1}{3}}-c^{\frac{1}{3}}=0$ or $(x^2+y^2-c^2)^3+27c^2x^2y^2=0$, here $m=6$, $n=4$, $\delta=4$, $\kappa=6$, $\iota=0$, $\tau=3$: the points I, J are each of them cusps, the tangents being the line IJ; the number of tangents from a cusp is $n-3$, $=1$, but for the cusp I or J, this tangent is the line IJ itself, so that we have $I=J=4$.

Theory when the Absolute is a conic.

When the Absolute is a conic the formulæ for the evolute are essentially the same as those in the former case, but the formulæ for the parallel curve are modified essentially and in a very remarkable manner. I observe that corresponding to a passage of the given curve through I or J we have a contact with the Absolute, so that in the present case f will properly denote the number of contacts of the given curve with the Absolute, and attending to this singularity only, viz. considering a given curve $(m, n, \delta, \kappa, \iota, \tau; \alpha, D)$ having f contacts with the Absolute, the formulæ for the evolute are

$$\begin{aligned} m'' &= \alpha - 3f, \\ n'' &= m+n - f, \\ \iota'' &= f, \\ \kappa'' &= -3m-3n+3\alpha-5f. \end{aligned}$$

In the case $f=0$, these at once follow from the two equations $n''=m+n$, and $\iota''=0$. The normal is the line joining a point of the given curve with the pole of the tangent; or, what is the same thing, it is the line joining the point of the given curve with the corresponding point of the reciprocal curve: the degrees of the two curves are m, n, and the correspondence is a $(1, 1)$ correspondence. Hence by the general theorem previously referred to, it follows that we have $n''=m+n$, and $\iota''=0$. Compare herewith the demonstration of the theorem in the case where the Absolute is a point-pair.

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The formulæ for the parallel curve are

$$\begin{aligned} m' &= 2m + 2n - 2f, \\ n' &= 2m + 2n - 2f, \\ \iota' &= \quad 2\alpha - 6f, \\ \kappa' &= \quad 2\alpha - 6f, \\ f' &= 2m + 2n - 2f, \end{aligned}$$

(so that $m' = n'$, $\iota' = \kappa'$). The intersections of the curve and Absolute are in this case the points f each twice, and besides $2m - 2f$ points; similarly the common tangents are the tangents at f each twice and besides $2n - 2f$ tangents. Now I remark that the parallel curve, when the radius of the variable circle is $= 0$, reduces itself to the original curve twice, together with the $2n - 2f$ common tangents, and the $2m - 2f$ common points; the order is thus $= 2m + (2n - 2f)$, and the class $= 2n + (2m - 2f)$: and these are the values in the general case where the radius of the variable circle is not $= 0$.

But in the remarkable case where the curve and its evolute have a (1, 2) correspondence, then I correct the formulæ by adding $-\Theta$ to the expressions for ι', κ' respectively. We have for the evolute of the parallel curve

$$\begin{aligned} m''' &= 2m'', \\ n''' &= 2n'', \\ \iota''' &= 2\iota'' + (2m + 2n - 4f) - \Theta, \\ \kappa''' &= 2\kappa'' + (2m + 2n - 4f) - \Theta, \end{aligned}$$

viz. assuming $\Theta = 2m + 2n - 4f$, this means that the evolute is the evolute of the original curve taken twice.

A very interesting case is when $m = n = f$: observe that neither $m - f$ nor $n - f$ can be negative, so that the assumed relation $m + n - 2f = 0$ would imply these two relations. We have here for the parallel curve $m' = 2m$, $n' = 2n$, $\iota' = 2\iota$, $\kappa' = 2\kappa$; the parallel curve in fact breaking up into two curves such as the given curve. And in this case the formulæ for the evolute assume the very simple form $m'' = \kappa$, $n'' = f$, $\iota'' = f$, $\kappa'' = -2f + 3\kappa$.

Whatever the original curve may be, we have for the parallel curve $m' = n' = f'$, so that the formulæ for the evolute of the parallel curve are of the foregoing form $m''' = \kappa'$, $n''' = f'$, $\iota''' = f' - \Theta$, $\kappa''' = -2f' + 3\kappa' - \Theta$, which agree with the above values of m''', n''', ι''', κ'''. In the particular case $m = n = f$,

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we have $\Theta = 0$, so that the evolute-formulæ, if originally written down without the terms in Θ, would still be $m''' = 2m''$, $n''' = 2n''$, $\iota''' = 2\iota''$, $\kappa''' = 2\kappa''$; viz. the evolute is here the original evolute taken twice; as already seen, the parallel curve consisted of two curves such as the original curve, and each of these has for its evolute the evolute of the original curve.

ON THE SPOKE ASYMPTOTES OF RHIZIC CURVES

By WILLIAM WALTON, M.A., Fellow of Trinity Hall.

PERHAPS the demonstration of the three following theorems may be interesting to students in the Differential Calculus:

(1) The asymptotes of each of two conjugate rhizic curves pass through a single point and divide plane space equiangularly like the spokes of a wheel.

(2) The common point of the asymptotes of either of two conjugate rhizic curves coincides with that of the asymptotes of the other rhizic curve.

(3) The collective system of asymptotes of both curves divide space equiangularly like spokes.

Let $$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n.$$

If $$f\{u + v\sqrt{(-1)}\} = P + Q\sqrt{(-1)},$$

where P and Q are possible functions of u and v, $P = 0$ and $Q = 0$ represent conjugate rhizic curves.

Let P_ν, Q_ν, denote the sum of all terms of ν dimensions in P, Q, respectively: then

$$p_{n-\nu}.\{u + v\sqrt{(-1)}\}^\nu = P_\nu + Q_\nu\sqrt{(-1)},$$
$$p_{n-\nu}.\{u - v\sqrt{(-1)}\}^\nu = P_\nu - Q_\nu\sqrt{(-1)}:$$

hence
$$\left.\begin{aligned} \frac{2P_\nu}{p_{n-\nu}} &= \{u + v\sqrt{(-1)}\}^\nu + \{u - v\sqrt{(-1)}\}^\nu, \\ \frac{2Q_\nu\sqrt{(-1)}}{p_{n-\nu}} &= \{u + v\sqrt{(-1)}\}^\nu - \{u - v\sqrt{(-1)}\}^\nu. \end{aligned}\right\} \ldots\ldots (1).$$

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The equations to the asymptotes of $P=0$ are, by Gregory's formulæ,* given by the equations

$$P_n = 0,$$

$$u' \frac{dP_n}{du} + v' \frac{dP_n}{dv} + P_{n-1} = 0.$$

Hence, by the formulæ (1), the asymptotes of the curve $P=0$ are determined by the equations

$$\{u + v\sqrt{(-1)}\}^n + \{u - v\sqrt{(-1)}\}^n = 0,$$

$$nu' [\{u + v\sqrt{(-1)}\}^{n-1} + \{u - v\sqrt{(-1)}\}^{n-1}]$$
$$+ nv'\sqrt{(-1)} [\{u + v\sqrt{(-1)}\}^{n-1} - \{u - v\sqrt{(-1)}\}^{n-1}]$$
$$+ p_1 [\{u + v\sqrt{(-1)}\}^{n-1} + \{u - v\sqrt{(-1)}\}^{n-1}] = 0;$$

and those of the curve $Q=0$ by the equations

$$\{u + v\sqrt{(-1)}\}^n - \{u - v\sqrt{(-1)}\}^n = 0,$$

$$\frac{nu'}{\sqrt{(-1)}} [\{u + v\sqrt{(-1)}\}^{n-1} - \{u - v\sqrt{(-1)}\}^{n-1}]$$
$$+ nv' [\{u + v\sqrt{(-1)}\}^{n-1} + \{u - v\sqrt{(-1)}\}^{n-1}]$$
$$+ \frac{p_1}{\sqrt{(-1)}} [\{u + v\sqrt{(-1)}\}^{n-1} - \{u - v\sqrt{(-1)}\}^{n-1}] = 0.$$

Putting $u = r\cos\theta$, $v = r\sin\theta$, the equations for the asymptotes of $P=0$ become

$$\cos n\theta = 0,$$

$$nu' \cos(n-1)\theta - nv' \sin(n-1)\theta + p_1 \cos(n-1)\theta = 0.$$

From these two equations we have

$$nu' \sin\theta - nv' \cos\theta + p_1 \sin\theta = 0,$$

and therefore, λ being an integer,

$$\theta = \frac{(2\lambda+1)\pi}{2n},$$

$$nv' = (nu' + p_1) \tan\frac{(2\lambda+1)\pi}{2n} \quad \ldots\ldots\ldots\ldots (2).$$

* *Cambridge Mathematical Journal*, Vol. IV., p. 42.

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In like manner the asymptotes of the curve $Q=0$ are determined by the equations

$$\sin n\theta = 0,$$

$$nu' \sin(n-1)\theta + nv' \cos(n-1)\theta + p_1 \sin(n-1)\theta = 0,$$

and therefore by the equations

$$\theta = \frac{\lambda\pi}{n},$$

$$nv' = (nu' + p_1) \tan\frac{\lambda\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots \quad (3).$$

From the equations (2) and (3) we see (1) that the asymptotes of the curve $P=0$ all pass through the point $u' = -\frac{p_1}{n}$, $v'=0$, and divide space equiangularly like the spokes of a carriage wheel, the inclinations of these asymptotes to the axis of u being

$$\frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \ldots \frac{(2n-1)\pi}{2n};$$

(2) that the asymptotes of the curve $Q=0$ all pass through the same point through which those of $P=0$ pass, and like them divide space equiangularly, their inclinations to the axis of u being

$$0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \ldots \frac{(n-1)\pi}{n};$$

(3) that also the collective system of the asymptotes of both curves divide space like the spokes of a wheel, the angle between any two successive spokes being $\frac{\pi}{2n}$.

December 3, 1870.

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THE DEDUCTION OF EULER'S EQUATIONS FROM THE LAGRANGE EQUATIONS.

By W. H. Besant, M.A.

Employing the ordinary figure (fig. 30), the general coordinates are θ, ϕ, and ψ, and the angular velocities about the principal axes A, B, and C, are given by the equations

$$\omega_1 = \theta' \sin\phi - \psi' \sin\theta \cos\phi,$$
$$\omega_2 = \theta' \cos\phi + \psi' \sin\theta \sin\phi,$$
$$\omega_3 = \phi' + \psi' \cos\theta.$$

Taking $2T$ for the *vis viva*,

$$2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2;$$

hence $$\frac{dT}{d\theta} = -A\omega_1\psi' \cos\theta \cos\phi + B\omega_2\psi' \cos\theta \sin\phi - C\omega_3\psi' \sin\theta,$$

$$\frac{dT}{d\theta'} = A\omega_1 \sin\phi + B\omega_2 \cos\phi,$$

and one of the Lagrange equations is

$$\frac{d}{dt}(A\omega_1 \sin\phi + B\omega_2 \cos\phi) + A\omega_1\psi' \cos\theta \cos\phi - B\omega_2\psi' \cos\theta \sin\phi + C\omega_3\psi' \sin\theta = \frac{dU}{d\theta} \quad \ldots\ldots\ldots\ldots (1),$$

U being the force function.

Observing that

$$\frac{dT}{d\phi} = A\omega_1(\theta' \cos\phi + \psi' \sin\theta \sin\phi) + B\omega_2(-\theta' \sin\phi + \psi' \sin\theta \cos\phi)$$
$$= A\omega_1\omega_2 - B\omega_1\omega_2,$$

the other equations are

$$\frac{d}{dt}.C\omega_3 + (B - A)\,\omega_1\omega_2 = \frac{dU}{d\phi} \quad \ldots\ldots\ldots\ldots (2),$$

$$\frac{d}{dt}(-A\omega_1 \sin\theta \cos\phi + B\omega_2 \sin\theta \sin\phi + C\omega_3 \cos\theta) = \frac{dU}{d\psi} \ldots (3).$$

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The equation (2) is itself one of Euler's equations; to obtain the others, arrange the equation (1) in the form

$$\sin\phi\left(\frac{d.A\omega_1}{dt}-B\omega_2\omega_3\right)+\cos\phi\left(\frac{d.B\omega_2}{dt}+A\omega_3\omega_1\right)$$
$$+C\omega_3\sin\theta.\psi'=\frac{dU}{d\theta}\ldots\ldots\ldots\ldots\ldots(4),$$

and equation (3) in the form

$$-\sin\theta\cos\phi\left(\frac{d.A\omega_1}{dt}-B\omega_2\omega_3\right)+\sin\theta\sin\phi\left(\frac{d.B\omega_2}{dt}+A\omega_3\omega_1\right)$$
$$+(B-A)\,\omega_1\omega_2\cos\theta+\cos\theta\,\frac{d.C\omega_3}{dt}-C\omega_3\theta'\sin\theta=\frac{dU}{d\psi},$$

which, by means of (2), becomes

$$-\sin\theta\cos\phi\left(\frac{d.A\omega_1}{dt}-B\omega_2\omega_3\right)+\sin\theta\sin\phi\left(\frac{d.B\omega_2}{dt}+A\omega_3\omega_1\right)$$
$$-C\omega_1\theta'\sin\theta=\frac{dU}{d\psi}-\cos\theta\,\frac{dU}{d\phi}\ldots\ldots\ldots\ldots\ldots(5).$$

From (4) and (5), we obtain

$$\sin\theta\left\{\frac{d.A\omega_1}{dt}-(B-C)\,\omega_2\omega_3\right\}$$
$$=\frac{dU}{d\phi}\sin\theta\sin\phi-\frac{dU}{d\psi}\cos\phi+\cos\theta\cos\phi\,\frac{dU}{d\phi}\ldots\ldots(6),$$

and $$\sin\theta\left\{\frac{d.B\omega_2}{dt}-(C-A)\,\omega_3\omega_1\right\}$$
$$=\frac{dU}{d\theta}\sin\theta\cos\phi+\frac{dU}{d\psi}\sin\phi-\cos\theta\sin\phi\,\frac{dU}{d\phi}\ldots\ldots(7).$$

Taking L, M, N, as the moments of forces about the principal axes, and resolving these moments about the axes C, F, and Z respectively, where F is the intersection of AB and XY,

$$\frac{dU}{d\phi}=N,\quad\frac{dU}{d\theta}=L\sin\phi+M\cos\phi,$$

$$\frac{dU}{d\psi}=-L\sin\theta\cos\phi+M\sin\theta\sin\phi+N\cos\theta.$$

From these equations we find that $L\sin\theta$, and $M\sin\theta$ are respectively equal to the right-hand members of equations (6) and (7), and hence Euler's equations follow at once.

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It is instructive to notice the interpretation of the equations (1), (2), and (3); they represent the equalities of the rates of change of the effective angular momenta about the axes of C, F, and Z, and of the moments of forces about those axes.

Thus the angular momentum about Z, is

$$-A\omega_1 \sin\theta \cos\phi + B\omega_2 \sin\theta \sin\phi + C\omega_3 \cos\theta,$$

from which the equation (3) follows at once, the axis Z being fixed, and $\frac{dU}{d\psi}$ the moment of forces about it.

For the equation (1), observe that in the fig. 30, K is the intersection of ZC with XY, and that F, the intersection of AB and XY, is the pole of ZC; and hence that $\frac{d\psi}{dt}$ is the rate of rotation of the axes K and F.

Let P and Q be the angular momenta about these axes, then

$$P = A\omega_1 \cos\theta \cos\phi - B\omega_2 \cos\theta \sin\phi + C\omega_3 \sin\theta,$$

and $$Q = A\omega_1 \sin\phi + B\omega_2 \cos\phi;$$

but the rate of change of the angular momentum about the moving axis F, is

$$\frac{dQ}{dt} + P\frac{d\psi}{dt};$$

hence, equating this to $\frac{dU}{d\theta}$, we obtain equation (1).

Lastly, the equation (2) follows from the expression for the rate of change of angular momentum about C, *i.e.*

$$\frac{d.C\omega_3}{dt} - A\omega_1.\omega_2 + B\omega_2.\omega_1,$$

which must be equated to $\frac{dU}{d\phi}$.

October, 1870.

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THE SMALL OSCILLATIONS OF A PARTICLE AND OF A RIGID BODY.*

By ROBERT STAWELL BALL, A.M, Professor of Applied Mathematics and Mechanism, Royal College of Science for Ireland.

I.

Introduction.

LAPLACE investigated the small oscillations of a particle on a sphere. Poisson solved a special case of the same problem on the ellipsoid. Lagrange discovered the general laws of small oscillations, and his methods have been improved by Messrs. Thomson and Tait. The results, of which the following is an abstract, have been obtained by an union of the method of Lagrange in its improved form, with some elegant theorems of kinematics discovered by M. Chasles. Demonstrations of some of the theorems here enunciated will be found in two papers written by the author.

"On the small oscillations of a particle on a surface under the action of any forces." *Quarterly Journal of Mathematics*, No. XXXIX. 1869.

"On the small oscillations of a rigid body about a fixed point under the action of any forces, and more particularly when gravity is the only force acting." *Transactions of the Royal Irish Academy*, Vol. XXIV. Science, Part XVI.

II.

A Particle.

1. There are in general three lines called *normal lines*, such that whatever be the small oscillations of a particle free in space, the movement is compounded of simple harmonic vibrations along the normal lines.

2. When the forces have a potential, a constant small quantity of energy would draw the particle along any radius vector from its position of rest to the surface of a certain ellipsoid, the normal lines are in the principal directions of this ellipsoid, and the lengths of the isochronous simple pendulum are proportional to the squares of its principal axes.

* Abstract of a paper read before the British Association at Liverpool.

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3. When the particle is constrained to a surface, the motion is compounded of vibrations in two directions on the surface, and when the forces have a potential, the tangent lines to these directions are at right angles.

III.

A free Rigid Body.

4. A free rigid body may receive any displacement by being screwed along an axis in space, the distance it travels along the axis when rotated through the angular unit being termed the pitch of the screw.

5. The movement of a free rigid body when making small oscillations is compounded of six normal movements, each consisting of a to and fro vibration about a *normal screw*, the position, pitch, and period of which depends upon the forces.

6. Whatever be the initial motion of the body, it may be distributed uniquely among the six normal screws, and thus the entire motion is determined.

IV.

A constrained Rigid Body.

7. If a rigid body have k degrees of freedom, its motion is compounded of vibrations about k normal screws.

8. A body capable of turning around a fixed axis, and sliding along it, has two degrees of freedom, its motion is compounded of that about two normal screws whose pitch is different, but both of which lie in the fixed axis.

9. A body, three points of which are limited to a plane, has three degrees of freedom. Its motion is compounded of vibrations about three normal screws whose pitch is zero, and the directions of which are perpendicular to the plane.

10. A body rotating about a fixed point has three degrees of freedom. Its motion is compounded of vibrations about three normal screws, whose pitch is zero, and whose directions pass through the point.

N.B. The screws in this case may be conveniently called the *normal axes*.

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V.

A rigid body rotating about a point, the forces having a potential.

11. The body may be moved from one position to any other position by rotation through a certain angle, about a certain axis passing through the point; this angle and axis are called the *axis of displacement* and the *angle of displacement* respectively.

12. On any axis through the point take a radius vector proportional to the small angular velocity, which a small quantity of energy would be able to communicate to the body about the axis. The quantity of energy being constant, the locus of this point on different axes is the well-known *momental ellipsoid.*

13. On any axis through the point take a radius vector proportional to the small angle, through which a small quantity of energy would be able to rotate the body about the axis from its position of equilibrium against the forces. The quantity of energy being constant, the locus of this point on different axes may be called the *ellipsoid of equal energy.*

14. The three common conjugate diameters of the momental ellipsoid and the ellipsoid of equal energy are the normal axes; the body would vibrate about either of these axes, as about a fixed axis, and its motion is always compounded of vibrations about these axes.

15. The length of the simple pendulum isochronous with the vibration about each normal axis is proportional to the square of the ratio of the corresponding diameter in the ellipsoid of equal energy to that of the momental ellipsoid.

16. The body is slightly disturbed from its position of rest by rotation about an axis of displacement, through an angle of displacement, and also by receiving a small angular velocity about an initial instantaneous axis. This displacement and velocity may be uniquely resolved into corresponding displacements and angular velocities about the normal axes, and thus the motion of the body is completely determined.

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VI.

A rigid body rotating round a fixed point, gravity being the only force acting.

17. A plane drawn in the momental ellipsoid conjugate to the vertical through the point of suspension is called the conjugate plane.

18. For small oscillations to be possible the instantaneous axis must lie initially in the conjugate plane, and it will continue there throughout the motion.

19. There are two normal axes which are thus constructed. Draw an ellipsoid whose axes are in the same directions as and proportional to the squares of those of the momental ellipsoid, the common conjugate diameters of the sections of these ellipsoids made by the conjugate plane are the normal axes.

20. The normal axes are not at right angles, except when the centre of gravity lies in one of the principal planes about the point of suspension; but a vertical plane drawn through one normal axis is always perpendicular to a vertical plane drawn through the other normal axis.

21. The body would vibrate about either of these normal axes, as about a fixed axis, and any small oscillation is compounded of simple vibrations about the normal axes.

Special attention is directed to the theorem of paragraph 19, which contains the solution of the conical pendulum under its most general form.

Royal College of Science for Ireland,
Dublin, *December 1st*, 1870.

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NOTE ON OSCULATING CURVES.

By Joseph Wolstenholme, M.A.

1. If *a curve S, similarly and similarly situated to a given curve A, have contact of the second order with a given curve B, at a point P, and C be the* centre (*or any point situated with respect to S, similarly to some fixed point with respect to A*), *PC will be the tangent at C to the locus of C.*

For if $OPQO'$ be contiguous points on the curve B, the curves through OPQ, PQO' similar and similarly situated to A, will in the limit be two curves S, S', or we may take the two curves as the limits of two curves touching B at P, and passing through contiguous points O, O' on opposite sides of P; but P will be a centre of similitude of these two curves, and P, C, C' will be on one straight line. Hence, in the limit when C, C' approach coincidence, PC will be a tangent at C to the locus of C.

2. *If a curve S, similar to a given curve A, in which the chord of curvature through the pole has a constant ratio to the radius vector, have contact of the third order with a given curve B at a point P, and C be the pole of S, then PC makes equal angles with the tangent at P and the tangent at C to the locus of C.*

For if $OPQRO'$ be five contiguous points on B, we can draw through $OPQR$, and through $PQRO'$ curves similar to A, and when P, Q, R move up to coincidence, these two curves will have the same circle of curvature at P, hence, if C, C' be their poles, $PC : PC' :: PL : PL'$, where PL, PL' are the corresponding chords of curvature. Hence, in the limit, when C, C' tend to coincidence, the tangent at C to the locus of C is parallel to the tangent at L to the circle of curvature, or makes with PC the same angle as the tangent at P. (These proofs of these two propositions are due to Mr. J. F. Moulton, Fellow of Christ's College).

The curve A is either an equiangular spiral, or one of the family $r^m = a^m \cos m\theta$, (the values ± 1 of m being excluded), but in the case of the equiangular spiral, the spiral

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of closest contact is drawn at each point, and is not similar to any fixed equiangular spiral. The family includes the parabola (focus pole); the rectangular hyperbola (centre); the cardioid (cusp); and the lemniscate of Bernoulli (node); and in all these cases the osculating curve is everywhere similar to itself, and the corresponding locus possesses the property here proved.

The curve for which the radius vector to the pole of the equiangular spiral of closest contact is constant, is the catenary of equal strength $y = a \log \sec \frac{x}{a}$, and in this case the arc described by C is always equal to the corresponding arc described by P. Many other interesting properties of the osculating spiral for different curves may be investigated, and the property here proved seems to be of much use in such questions.

EXAMPLE OF A SPECIAL DISCRIMINANT.

By Prof. CAYLEY.

IF we have a function $(a, \ldots)(x, y, z)^n$, where the coefficients $(a, \ldots)$ are such that the curve $(a, \ldots)(x, y, z)^n = 0$ has a node, and *à fortiori* if this curve has any number of nodes or cusps, the discriminant of the function (that is, the discriminant of the general function $(*)(x, y, z)^n$, substituting in such discriminant for the coefficients their values for the particular function in question) vanishes *identically*. But the particular function has nevertheless a *special discriminant*, viz., this is a function of the coefficients which, equated to zero, gives the condition that the curve may have (besides the nodes or cusps which it originally possesses) one more node; and the determination of this special discriminant (which, observe, is not deducible from the expression of the discriminant of the general function $(*)(x, y, z)^n$) is an interesting problem. I have, elsewhere, shown that if the curve in question $(a, \ldots)(x, y, z)^n = 0$ has δ nodes and κ cusps, then the degree of the special discriminant in regard to the coefficients a, &c., of the function is $= 3(n-1)^2 - 7\delta - 11\kappa$: and I propose to verify this in the case of a quartic curve with two cusps.

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Consider the curve

$$6nx^2y^2 + 12rz^2xy + (4gx + 4iy + cz)\,z^3 = 0,$$

where $x=0$ is the tangent at a cusp; $y=0$ the tangent at a cusp; and $z=0$ the line joining the two cusps.

For the special discriminant we have

$$3nxy^2 + 3ryz^2 + gz^3 = 0,$$

$$3nx^2y + 3rxz^2 + iz^3 = 0,$$

$$z\,\{6rxy + (3gx + 3iy + 4cz)\,z\} = 0;$$

the last of which may be replaced by the equation of the curve.

Assume $x = \lambda z$, $y = \mu z$, the first two equations give

$$3\,(n\lambda\mu + r)\,\mu + g = 0,$$

$$3\,(n\lambda\mu + r)\,\lambda + i = 0,$$

whence also

$$6n\lambda^2\mu^2 + 6r\lambda\mu + g\lambda + i\mu = 0,$$

and the equation of the curve gives

$$6n\lambda^2\mu^2 + 12r\lambda\mu + 4g\lambda + 4i\mu + c = 0,$$

whence eliminating $g\lambda + i\mu$ we find

$$18n\lambda^2\mu^2 + 12r\lambda\mu - c = 0.$$

Moreover the first two equations give

$$9\,(n\lambda\mu - r)^2\,\lambda\mu - ig = 0,$$

or putting $\lambda\mu = \theta$ we have

$$18n\theta^2 + 12r\theta - c = 0,$$

$$9\,(n\theta + r)^2\,\theta - ig = 0,$$

from which θ is to be eliminated.

The equations are

$$18n\theta^2 + 12r\theta - c = 0,$$

$$9n^2\theta^3 + 18nr\theta^2 + 9r^2\theta - ig = 0,$$

and thence

$$18n^2\theta^3 + 36nr\theta^2 + 18r^2\theta - 2ig = 0,$$

$$18n^2\theta^3 + 12nr\theta^2 - cn\theta = 0,$$

$$24nr\theta^2 + (18r^2 + cn)\,\theta - 2ig = 0,$$

$$18nr\theta^2 + 12r^2\theta - cr = 0,$$

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$$(6r^2 + 3cn)\,\theta - 6ig + 4cr = 0,$$

$$\theta = \frac{6ig - 4cr}{6r^2 + 3cn} = \frac{2}{3}\,\frac{3ig - 2cr}{2r^2 + cn};$$

or substituting in $18n\theta^2 + 12r\theta - c = 0$, this is

$$8n\,(3ig - 2cr)^2 + 8r\,(3ig - 2cr)\,(2r^2 + cn) - c\,(2r^2 + cn)^2 = 0.$$

Hence developing, the special discriminant is

$$\begin{aligned} \square = &-1 \quad c^3n^2 \\ &+12 \quad c^2nr^2 \\ &-72 \quad cginr \\ &-36 \quad cr^4 \\ &+72 \quad g^2i^2n \\ &+48 \quad gir^3, \end{aligned}$$

which is as it should be of the degree $5, = 3.3^2 - 11.2$.

ON THE ALGEBRA OF MAGIC SQUARES. No. III.

By Joseph Horner, M.A., Clare College.

Having already discussed the various classes of odd magic squares connected with the form $R_N ax + by$, we now proceed to even squares. In doing this, we shall represent a series of unities by their signs only, and in this sense speak of multiplying by those signs.

Take in any order the zero arithmetical series of $2n$ elements, $-2n-1, -2n+1, \ldots 2n-1, 2n+1$, and let λ and $-\lambda$ represent any conjugate pair of these elements. If λ and $-\lambda$ be each multiplied by one and the same series of $2n$ signs, and a like operation extended to all the values which λ can take, and the results be arranged in columns

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according to the assumed order of the original series, we manifestly form a square, every row of which is a zero series.

We have next to consider what must be the nature of these series, in order that the diagonals and columns of the square may be zero series. Here simplicity will be promoted by taking axes through the centre of the square parallel to its sides, and numbering the distances of the rows and columns from the centre positively and negatively on these coordinates as in the Cartesian system. Then (x, y) will denote a point at the intersecton of the x^{th} row and y^{th} column. Suppose then that the conjugate columns headed with λ and $-\lambda$ are the r^{th} and s^{th}. The grave diagonal will meet these at the points r and s respectively, reckoned from the axis of x. If, therefore, maintaining the principle already laid down for the zero rows, we multiply λ and $-\lambda$ by the same sign (which we may express by the optional sign a), at the points r and s of both columns, λ and $\overline{\lambda}$ will both appear in the grave diagonal; and when a similar operation has been extended to all the values that λ can take, the grave diagonal will be a zero series. Again, the acute diagonal meets the same columns at $-r$ and $-s$ respectively; and if at these points we multiply by the optional sign b, and apply the like process to all the values of λ, the acute diagonal will be a zero series. We may now so choose the signs a and b, and those for the remaining $2(n-2)$ vacant places of each pair of conjugate columns, that n of them shall be positive and n negative. Thus the conjugate columns of λ's are made zero series; and when this has been done for every value of λ, the zero square is completed. If, however, for any particular value of λ, the elements are equidistant from the extremities of the series in its assumed order, *i.e.* if $r+s=0$, then $-s=r$, and $-r=s$; so that the columns are symmetrically situated and the two sets of signs a, b, coalesce into one set. In this case we can choose the sign a and those in the $2(n-1)$ remaining places in the same manner as before stated. It will be convenient to keep the sign-columns separate from the terms of the arithmetical series, and to speak of this system as the sign-square. If $n=2$ or 3, it is manifest that a and b must represent opposite signs. Also, if n be even, the $2(n-2)$ or $2(n-1)$ vacant places can always be filled up, if desirable, with pairs of like signs; but if n be odd there must be at least one pair of unlike signs in every column. In the following examples each letter represents an optional sign. The dots stand

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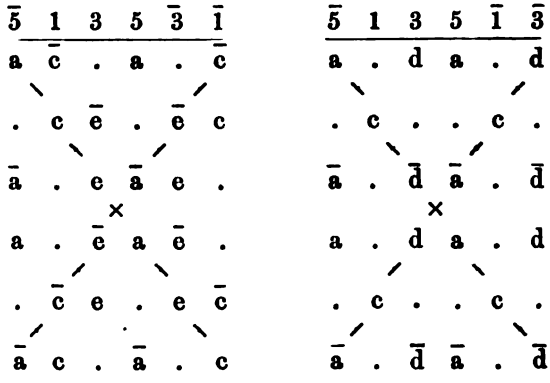
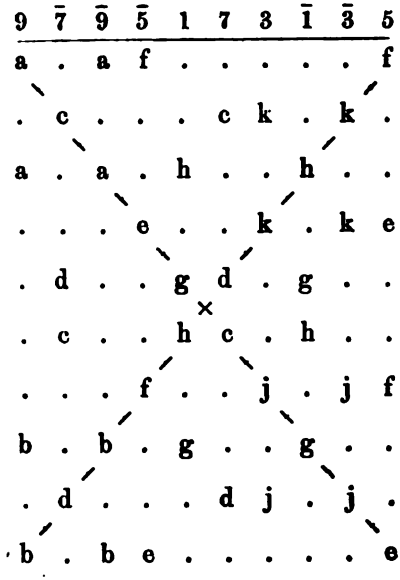
vacant places which are to be afterwards filled up with suitable signs:





It will be seen that, while we keep the conjugate columns of the sign-square unaltered, the numbers, 1, 3, 5, ... may be interchanged in any manner.

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Also, reversing the order of all the columns would cause no change in any of them as to signs.

We now go on to enquire how and under what conditions two such zero squares can be combined by juxta-position into a magic square.

Suppose λ and $-\lambda$ to stand at the head of two columns in the first square, and μ and $-\mu$ of two in the second. Turn the second square so that its rows become columns; in particular, so that the upper row becomes the left-hand column. When the squares now have their respective elements in juxta-position, suppose that the four permutations in which λ precedes, standing thus:

	λ	$\ldots\bar{\lambda}$
	$\vdots$	
μ	$\ldots\lambda_1\mu_1$	$\bar{\lambda}_1\mu_2$
$\vdots$		
μ	$\bar{\lambda}_2\mu_1$	$\bar{\lambda}_2\bar{\mu}_2$

where λ_1, λ_2; μ_1, μ_2 are numerically λ and μ, but with signs at present unsettled. First imagine λ_1, λ_2 to result from multiplying λ by unlike signs, and μ_1, μ_2 from multiplying μ by unlike signs. Whence $\lambda_2 = -\lambda_1$ and $\mu_2 = -\mu_1$; so that the four pairs become $\lambda_1\mu_1$, $\bar{\lambda}_1\mu_1$, $\bar{\lambda}_1\bar{\mu}_1$, $\lambda_1\bar{\mu}_1$, which are not all different, whatever the signs belonging to λ_1 and μ_1. Next conceive λ_1, λ_2 to result from λ and μ_1, μ_2 from μ by multiplying in each instance by like signs. Then $\lambda_2 = \lambda_1$ and $\mu_2 = \mu_1$; and we have the four pairs $\lambda_1\mu_1$, $\bar{\lambda}_1\mu_1$, $\lambda_1\bar{\mu}_1$, $\bar{\lambda}_1\bar{\mu}_1$; which are all different. Thirdly, if either pair, as $\lambda_1\lambda_2$, result from multiplying λ by like, and the other $\mu_1\mu_2$ from multiply μ by unlike signs, we have $\lambda_2 = \lambda_1$ and $\mu_2 = -\mu_1$, and accordingly $\lambda_1\mu_1$, $\bar{\lambda}_1\bar{\mu}_1$, $\lambda_1\bar{\mu}_1$, $\bar{\lambda}_1\mu_1$ all different. Upon the whole, therefore, we shall always obtain the four permutations in which λ stands first, provided λ and μ are not both multiplied by unlike signs in the two places where they stand in their respective columns. And this holds good when λ and μ are equal.

When λ and μ are unequal, the juxta-posed squares will contain four other permutations of λ and μ having μ first, to which the same law is applicable; and the eight make up all the permutations of the elements $\lambda, -\lambda, \mu, -\mu$, which are required to be found in the magic square. When λ and μ are equal, they can only have the four permutations above mentioned.

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Now assume that the λ and $-\lambda$ columns in square I. have the situations r and s, and the μ and $-\mu$ columns in square II. the situations ρ and σ. Then when square II. is turned as aforesaid, μ and $-\mu$ are in the rows ρ and σ. Hence the permutations $\lambda\mu$, $\lambda\bar{\mu}$, $\bar{\lambda}\mu$ have the positions (r, ρ), (r, σ), (s, ρ); and if λ has unlike signs at (r, ρ) and (r, σ), μ must have like signs at (r, ρ) and (s, ρ). That is, referred to the original position of square II., μ must have like signs at $(\rho, \bar{r})$ and $(\rho, \bar{s})$. As this law depends not on the values of λ and μ, but solely on the positions of their conjugate columns, we may take *Irs* generally as an abridged form of the words, 'each column of that conjugate pair in the sign-square I.,' which are at the distances r and s from the centre; and then the rule will be conveniently expressed as follows:

If *Irs* has unlike signs at the points ρ, σ, *II*$\rho\sigma$ must have like signs at points $\bar{r}$, $\bar{s}$. And *vice versâ*, if *II*$\rho\sigma$ has unlike signs at the points $\bar{r}$, $\bar{s}$, *Irs* must have like signs at the points ρ, σ.

If, however, square II. be turned so that its upper row becomes the right-hand column, then if *Irs* has unlike signs at ρ, σ, *II*$\rho\sigma$ must have like signs at r, s; and *vice versâ*. N.B. Square II. is the displaced square. It will be sufficient for theory to treat the former of these cases.

If *Irs*, *Ir's'*, ... have unlike signs at the same points ρ, σ in their respective columns, *II*$\rho\sigma$ must have like signs at $\bar{r}$, $\bar{s}$, at $\bar{r}'$, $\bar{s}'$, ... Consequently when n is odd all the n sets *Irs*, *Ir's'*, ... cannot have unlike signs in the same rows ρ, σ; otherwise *II*$\rho\sigma$ would contain n pairs of like signs, which is impossible.

When r and s are numerically unequal, *Irs* has of course a pair of like signs at r, s and another pair at $\bar{r}$, $\bar{s}$; and the same may be said of *II*$\rho\sigma$. See the examples above.

The conditions for a magic square are manifestly irreconcileable if $n = 1$. If $n = 2$ or 3, the zero squares have peculiarities which render many combinations of them impossible.

First suppose $n = 2$. Represent *any* two numerically unequal terms of the series $\bar{2}\ \bar{1}\ 1\ 2$ by r, s; then $\bar{r}$, $\bar{s}$ represent the remaining two. There are three forms of zero squares, having their conjugate columns at r, s; $\bar{r}$, $\bar{s}$, or r, $\bar{s}$; $\bar{r}$, s, or r, $\bar{r}$; s, $\bar{s}$: the two first differing *inter se*, but being of the same type with reference to the third.

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Now we cannot have such associations as Irs and $II\bar{r}s$ or Irs and $IIs\bar{s}$. For Irs has contrary signs at $r, \bar{s}$, and therefore $II\bar{r}s$ must have the same sign at r, s. But it has the same sign at $\bar{r}, s$, and therefore a common sign at $r, \bar{r}, s$; which is impossible. In like manner $IIs\bar{s}$ would require a common sign at $r, s, \bar{s}$; which is impossible. There only remain associations of the two kinds $Irs, \bar{r}\,\bar{s}$ with $II\bar{r}s, \bar{r}\,\bar{s}$, and $Ir\bar{r}, s\bar{s}$ with $II\bar{r}\bar{r}, s\bar{s}$, where we use such an expression as $Irs, \bar{r}\,\bar{s}$ to denote 'the square I having its conjugate columns at r, s and $\bar{r}, \bar{s}$.' Both these kinds are possible and include $I12, \bar{1}\,\bar{2}$ combined with $II12, \bar{1}\,\bar{2}$, $I1\,\bar{2}, \bar{1}\,2$ with $II1\bar{2}, \bar{1}\,2$ and $I1\bar{1}, 2\bar{2}$ with $II\bar{1}1, 2\bar{2}$.

Next assume $n=3$, and let r, s, t represent any three numerically different terms of the series $\bar{3}\,\bar{2}\,\bar{1}\,1\,2\,3$. It is manifest that Irs has a pair of like signs at r, s, a pair of the opposite like signs at $\bar{r}, \bar{s}$, and opposite signs at $t, \bar{t}$. Consequently Irs has opposite signs at $r, \bar{r}$, at $s, \bar{s}$ and at $t, \bar{t}$; and this, whatever r and s be. Hence a square of the type $Irs, t\bar{r}, \bar{s}\,\bar{t}$, in which none of the conjugate columns are symmetrically situated, cannot be associated with another which contains such a form as $II\rho\bar{\rho}$; that is, in which is a conjugate pair symmetrically situated. For whether ρ represent r, s or t, each of the sets $Irs, It\bar{r}, I\bar{s}\,\bar{t}$ has unlike signs at $\rho, \bar{\rho}$, and consequently $II\rho\bar{\rho}$ would have like signs at $\bar{r}, \bar{s}$, at $\bar{t}, r$ and at s, t; three pairs, which is impossible. Again, such associations as Irs with $II\bar{r}\bar{s}$ are impossible, because Irs has unlike signs at $\bar{r}, s$, and therefore $II\bar{r}\bar{s}$ would require like signs at r, s, besides having like signs at $r, \bar{s}$ and at $\bar{r}, s$. That is $II\bar{r}\bar{s}$ would have one sign at $r, \bar{r}, s, \bar{s}$; which is impossible.

Now Square I has three types of conjugate columns, viz.:

(a) comprising 3 wholly unsymmetrical sorts different *inter se*, $Irs, \bar{t}\,\bar{r}, \bar{s}\,\bar{t}$ or $I\bar{r}\,\bar{s}, \bar{t}r, \bar{s}t$; $Irt, \bar{r}\,s, \bar{s}\,\bar{t}$ and $Irs, \bar{t}\,\bar{r}, s\bar{t}$;

(b) comprising 3 partially symmetrical sorts, $Irs, \bar{r}\,\bar{s}, t\,\bar{t}$; $Irt, r\,\bar{t}, s\,\bar{s}$ and $Irs, \bar{r}s, t\bar{t}$;

(c) containing one wholly symmetrical sort $I\,r\,\bar{r}, s\,\bar{s}, t\,\bar{t}$. The same is true of Square II.

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Here for the reasons already given, no two squares of different types are compatible with each other. Neither are two of different sorts of the type (*a*). It may likewise be proved that two of different sorts of the type (*b*) are incompatible. For, both *Irs* and $I\,\bar{r}\,\bar{s}$ having unlike signs at $s, \bar{s}$, $II\bar{s}\bar{s}$ would require like signs at $\bar{r}, \bar{s}$ and at r, s, having already like signs at $s, \bar{s}$; and so have one sign at $r, s, \bar{r}, \bar{s}$; which is impossible. The same reasoning applies to $Ir\bar{s}$ and to $I\bar{r}s$ in combination with $II\bar{s}\bar{s}$.

It remains therefore to enquire under what conditions squares of the same sort under each type may be combined.

Type (*a*). $Irs, t\,\bar{r}, \bar{s}\,\bar{t}$ with $IIrs, t\,\bar{r}, \bar{s}\,\bar{t}$. Here *Irs* must have unlike signs either at $t\,\bar{r}$ or at $\bar{s}, \bar{t}$; and $It\bar{r}$ must, either at r, s or at $\bar{s}, \bar{t}$. But they cannot both have unlike signs at $\bar{s}, \bar{t}$ in their respective columns, since $II\,\bar{s}\,\bar{t}$ would then require like signs at $\bar{r}, \bar{s}$ and at $\bar{t}, r$, having already like signs at s, t and at $\bar{s}, \bar{t}$; and consequently require like signs at $\bar{r}, \bar{s}, \bar{t}$, and therefore at $r, \bar{r}$. So that $II\,\bar{s}\,\bar{t}$ would have 3 pairs of like signs, viz. at s, t, at $\bar{s}, \bar{t}$ and at $r, \bar{r}$; which is impossible.

Suppose then *Irs* to have unlike signs at $t, \bar{r}$, $I\,t\,\bar{r}$ at $\bar{s}, \bar{t}$, and $I\,\bar{s}\,\bar{t}$ at r, s. Hence $IIrs$ has like signs at s, t, having them already at $\bar{r}, \bar{s}$; and consequently unlike signs at $t, \bar{r}$. In like manner it may be shewn, that $II\,t\,\bar{r}$ has unlike signs at $\bar{s}, \bar{t}$, and $II\,\bar{s}\,\bar{t}$ at r, s. In other words, the squares Irs $t\,\bar{r}, \bar{s}\,\bar{t}$ must have unlike signs at the same pairs of points in their respective columns. That is, one scheme must represent them both. The same reasoning applies if one of the squares, as the first, have all its columns in reverse order, viz. $I\,\bar{r}\,\bar{s}, \bar{t}\,r, st$.

Under type (*b*) we have only to examine the combination of $Irs, \bar{r}\,\bar{s}, t\,\bar{t}$ with $IIrs, \bar{r}\,\bar{s}, \bar{t}\,\bar{t}$. Here *Irs* and $I\,\bar{r}\,\bar{s}$ have each unlike signs at $t, \bar{t}$; so that $II\,t\,\bar{t}$ would require like signs at $\bar{r}, \bar{s}$, as well as at $t, \bar{t}$; which is impossible. Hence there are no possible cases of this type.

Squares of the type (*c*) can be combined, provided no pair, as $I\,r\,\bar{r}$ and $I\,s\,\bar{s}$, have each unlike signs at $t, \bar{t}$; for then $II\,t\,\bar{t}$ would require like signs at $r, \bar{r}$, at $s, \bar{s}$ and at $t, \bar{t}$; which is impossible. The same applies, *vice versâ*, to square II.

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Example of the combination

$$Irs, t\bar{r}, \bar{s}\bar{t} \text{ and } IIrs, t\bar{r}, \bar{s}\bar{t}.$$

Take $r=1$, $s=2$, $t=3$. The Roman letters represent the optional signs employed in forming the diagonals; the Italic those afterwards supplied.

c	c	b	a	a	b
c	c	$\bar{b}$	a	a	$\bar{b}$
$\bar{c}$	$\bar{c}$	$\bar{\text{b}}$	a	a	$\bar{\text{b}}$
c	c	b	$\bar{a}$	$\bar{a}$	b
$\bar{\text{c}}$	$\bar{\text{c}}$	b	$\bar{\text{a}}$	$\bar{\text{a}}$	b
$\bar{\text{c}}$	$\bar{\text{c}}$	$\bar{\text{b}}$	$\bar{a}$	$\bar{a}$	$\bar{\text{b}}$

This is the scheme common to both squares; but it is not necessary to give a, b, c the same values in each square. For the first square put $-$ for a, $+$ for b, $-$ for c; and for the second put $+$ for a, $-$ for b, and $-$ for c. Then the two sign-squares are

$-$	$-$	$+$	$-$	$-$	$+$	and	$-$	$-$	$-$	$+$	$+$	$-$
$-$	$-$	$-$	$-$	$-$	$-$		$-$	$-$	$+$	$+$	$+$	$+$
$+$	$+$	$-$	$-$	$-$	$-$		$+$	$+$	$+$	$+$	$+$	$+$
$-$	$-$	$+$	$+$	$+$	$+$		$-$	$-$	$-$	$-$	$-$	$-$
$+$	$+$	$+$	$+$	$+$	$+$		$+$	$+$	$-$	$-$	$-$	$-$
$+$	$+$	$-$	$+$	$+$	$-$		$+$	$+$	$+$	$-$	$-$	$+$

Hence applying arbitrarily the multipliers $5, \bar{5}$; $3, \bar{3}$; $1, \bar{1}$ to the several pairs of conjugate columns, we obtain such squares as

$\bar{5}$	5	1	$\bar{3}$	3	$\bar{1}$	and	$\bar{3}$	3	5	$\bar{1}$	1	$\bar{5}$
$\bar{5}$	5	$\bar{1}$	$\bar{3}$	3	1		$\bar{3}$	3	$\bar{5}$	$\bar{1}$	1	5
5	$\bar{5}$	$\bar{1}$	$\bar{3}$	3	1		3	$\bar{3}$	$\bar{5}$	$\bar{1}$	1	5
$\bar{5}$	5	1	3	$\bar{3}$	$\bar{1}$		$\bar{3}$	3	5	1	$\bar{1}$	$\bar{5}$
5	$\bar{5}$	1	3	$\bar{3}$	$\bar{1}$		3	$\bar{3}$	5	1	$\bar{1}$	$\bar{5}$
5	$\bar{5}$	$\bar{1}$	3	$\bar{3}$	1		3	$\bar{3}$	$\bar{5}$	1	$\bar{1}$	5

in each of which $\mp 5, \mp 3, \pm 1$ may be permuted in all possible ways. So that turning the second square, we have the magic squares

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$\bar{5}\,\bar{5}$	$5\,5$	$1\,5$	$\bar{3}\,\bar{5}$	$3\,\bar{5}$	$\bar{1}\,5$		$5\,\bar{5}$	$5\,5$	$5\,1$	$5\,\bar{3}$	$\bar{5}\,3$	$5\,\bar{1}$
$\bar{5}\,1$	$5\,1$	$\bar{1}\,1$	$\bar{3}\,1$	$3\,\bar{1}$	$1\,\bar{1}$		$1\,\bar{5}$	$1\,5$	$1\,\bar{1}$	$\bar{1}\,\bar{3}$	$1\,3$	$\bar{1}\,1$
$5\,\bar{1}$	$\bar{5}\,\bar{1}$	$\bar{1}\,\bar{1}$	$3\,1$	$3\,1$	$1\,1$		$\bar{1}\,5$	$\bar{1}\,\bar{5}$	$\bar{1}\,\bar{1}$	$1\,\bar{3}$	$1\,3$	$1\,1$
$\bar{5}\,5$	$5\,5$	$1\,\bar{5}$	$3\,5$	$\bar{3}\,5$	$\bar{1}\,\bar{5}$		$5\,\bar{5}$	$\bar{5}\,5$	$\bar{5}\,1$	$5\,3$	$5\,\bar{3}$	$\bar{5}\,\bar{1}$
$5\,3$	$\bar{5}\,3$	$1\,\bar{3}$	$3\,3$	$\bar{3}\,3$	$\bar{1}\,3$		$3\,5$	$3\,\bar{5}$	$\bar{3}\,1$	$3\,3$	$\bar{3}\,\bar{3}$	$\bar{3}\,\bar{1}$
$5\,\bar{3}$	$\bar{5}\,\bar{3}$	$\bar{1}\,3$	$3\,\bar{3}$	$\bar{3}\,3$	$1\,3$		$\bar{3}\,5$	$\bar{3}\,\bar{5}$	$3\,\bar{1}$	$\bar{3}\,3$	$3\,\bar{3}$	$3\,1$

which, by substituting the natural series for the zero arithmetical progression, give

1	36	24	7	25	18	and	1	36	34	2	5	33
4	34	16	9	27	21		19	24	21	14	17	16
33	3	15	10	28	22		18	13	15	20	23	22
6	31	19	30	12	13		31	6	4	35	32	3
35	5	20	29	8	14		30	25	10	29	8	9
32	2	17	26	11	23		12	7	27	11	26	28

Example of

Irs, $t\bar{r}$, $\bar{s}\,\bar{t}$ with *II*$\bar{r}\,\bar{s}$, $\bar{t}\,r$, *st*.

Taking for the first square the same scheme as before, that for the second becomes

b	*a*	*a*	b	c	c
$\bar{b}$	a	a	$\bar{b}$	c	c
$\bar{\text{b}}$	a	a	$\bar{\text{b}}$	$\bar{c}$	$\bar{c}$
b	$\bar{\text{a}}$	$\bar{\text{a}}$	b	*c*	*c*
b	$\bar{\text{a}}$	$\bar{\text{a}}$	*b*	$\bar{\text{c}}$	$\bar{\text{c}}$
$\bar{\text{b}}$	$\bar{a}$	$\bar{a}$	$\bar{\text{b}}$	$\bar{\text{c}}$	$\bar{\text{c}}$

In the first write + for *a*, + for *b*, − for *c*; and in the second − for *a*, + for *b*, − for *c*, and apply the multipliers arbitrarily as before, and we have the squares

$\bar{5}$	5	$\bar{1}$	3	$\bar{3}$	1	and	5	$\bar{1}$	1	$\bar{5}$	$\bar{3}$	3
$\bar{5}$	5	1	3	$\bar{3}$	$\bar{1}$		$\bar{5}$	$\bar{1}$	1	5	$\bar{3}$	3
5	$\bar{5}$	1	3	$\bar{3}$	$\bar{1}$		$\bar{5}$	$\bar{1}$	1	5	3	$\bar{3}$
$\bar{5}$	5	$\bar{1}$	$\bar{3}$	3	1		5	1	$\bar{1}$	$\bar{5}$	$\bar{3}$	3
5	$\bar{5}$	$\bar{1}$	$\bar{3}$	3	1		5	1	$\bar{1}$	$\bar{5}$	3	$\bar{3}$
5	$\bar{5}$	1	$\bar{3}$	3	$\bar{1}$		$\bar{5}$	1	$\bar{1}$	5	3	$\bar{3}$

where ± 5, ± 3, ± 1, admit of all possible permutations.

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The combined magic squares are

$\bar{5}\,3$	$5\,3$	$\bar{1}\,\bar{3}$	$3\,3$	$\bar{3}\,\bar{3}$	$1\,\bar{3}$	and	$3\,\bar{5}$	$3\,5$	$\bar{3}\,\bar{1}$	$3\,3$	$\bar{3}\,\bar{3}$	$3\,\bar{1}$
$\bar{5}\,\bar{3}$	$5\,\bar{3}$	$1\,3$	$3\,\bar{3}$	$\bar{3}\,3$	$\bar{1}\,3$		$\bar{3}\,\bar{5}$	$\bar{3}\,5$	$3\,1$	$\bar{3}\,3$	$3\,\bar{3}$	$3\,\bar{1}$
$5\,\bar{5}$	$\bar{5}\,5$	$1\,5$	$3\,\bar{5}$	$\bar{3}\,\bar{5}$	$\bar{1}\,5$		$\bar{5}\,5$	$5\,\bar{5}$	$5\,1$	$\bar{5}\,3$	$\bar{5}\,\bar{3}$	$5\,\bar{1}$
$\bar{5}\,1$	$5\,1$	$\bar{1}\,1$	$\bar{3}\,\bar{1}$	$3\,\bar{1}$	$1\,\bar{1}$		$1\,\bar{5}$	$1\,5$	$1\,\bar{1}$	$\bar{1}\,\bar{3}$	$1\,\bar{3}$	$\bar{1}\,1$
$5\,\bar{1}$	$\bar{5}\,\bar{1}$	$\bar{1}\,\bar{1}$	$\bar{3}\,1$	$3\,1$	$1\,1$		$\bar{1}\,5$	$\bar{1}\,\bar{5}$	$\bar{1}\,\bar{1}$	$1\,\bar{3}$	$1\,3$	$1\,1$
$5\,5$	$\bar{5}\,\bar{5}$	$1\,\bar{5}$	$3\,5$	$3\,5$	$\bar{1}\,\bar{5}$		$5\,5$	$\bar{5}\,\bar{5}$	$\bar{5}\,1$	$5\,\bar{3}$	$5\,3$	$\bar{5}\,\bar{1}$

which give, by substituting the series of natural numbers for the zero progression,

5	35	14	29	8	20	and	25	30	9	29	8	10
2	32	23	26	11	17		7	12	28	11	26	27
31	6	24	25	7	18		6	31	34	5	2	33
4	34	16	9	27	21		19	24	21	14	17	16
33	3	15	10	28	22		18	13	15	20	23	22
36	1	19	12	30	13		36	1	4	32	35	3

When n exceed 3, so many of the signs are optional that particular rules are little necessary and in fact become scarcely possible. Obviously the simplest squares are those in which the conjugate columns are symmetrically situated, and the case of n even is simpler than that of n odd. In comparing the two squares it will conduce to clearness if the columns be arranged separately as seen below, and when filled up be re-composed into a square:

I.					II.			
$\bar{4}\,\bar{1}$	$\bar{3}\,2$	$\bar{2}\,4$	$3\,1$		$\bar{4}\,2$	$\bar{3}\,\bar{1}$	$\bar{2}\,1$	$3\,4$
b	d	e	g	...4...	k	m	n	q
b	d	e	g	...3...	k	m	n	q
a	c	f	h	...2...	j	l	p	r
b	d	e	g	...1...	k	m	n	q
a	c	f	h	...$\bar{1}$...	j	l	p	r
b	d	e	g	...$\bar{2}$...	k	m	n	q
a	c	f	h	...$\bar{3}$...	j	l	p	r
a	c	f	h	...$\bar{4}$...	j	l	p	r

where we have assumed that a and b; c and d; ... are opposite signs. Here, in completing square I., we attend to

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the pairs of rows at 4, $\bar{2}$; 3, 1; 2, $\bar{1}$; $\bar{3}$, $\bar{4}$; and in completing square II., to the pairs at 4, 1; 3, $\bar{2}$; 2, $\bar{4}$; 3, $\bar{1}$. From the above we may derive such squares as

7	5	3	$\bar{7}$	1	$\bar{5}$	$\bar{1}$	$\bar{3}$
+	−	+	+	+	−	+	+
+	−	+	+	+	−	+	+
−	+	−	−	−	+	−	−
+	−	+	+	+	−	+	+
−	+	−	−	−	+	−	−
+	−	+	+	+	−	+	+
−	+	−	−	−	+	−	−
−	+	−	−	−	+	−	−

7	$\bar{5}$	3	5	$\bar{3}$	$\bar{7}$	1	$\bar{1}$
+	−	+	−	+	+	+	+
+	−	+	−	+	+	+	+
−	+	−	+	−	−	−	−
+	−	+	−	+	+	+	+
−	+	−	+	−	−	−	−
+	−	+	−	+	+	+	+
−	+	−	+	−	−	−	−
−	+	−	+	−	−	−	−

and thence

$7\,\bar{1}$	$\bar{5}\,\bar{1}$	$3\,1$	$\bar{7}\,\bar{1}$	$1\,1$	$5\,\bar{1}$	$\bar{1}\,1$	$\bar{3}\,1$
$7\,1$	$\bar{5}\,1$	$3\,\bar{1}$	$\bar{7}\,1$	$1\,\bar{1}$	$5\,1$	$\bar{1}\,\bar{1}$	$\bar{3}\,\bar{1}$
$7\,7$	$5\,\bar{7}$	$\bar{3}\,7$	$7\,\bar{7}$	$\bar{1}\,7$	$\bar{5}\,\bar{7}$	$1\,7$	$3\,7$
$7\,\bar{3}$	$\bar{5}\,\bar{3}$	$3\,\bar{3}$	$\bar{7}\,\bar{3}$	$1\,\bar{3}$	$5\,\bar{3}$	$\bar{1}\,\bar{3}$	$\bar{3}\,\bar{3}$
$\bar{7}\,5$	$5\,5$	$\bar{3}\,5$	$7\,5$	$\bar{1}\,5$	$\bar{5}\,5$	$1\,5$	$3\,5$
$7\,3$	$\bar{5}\,3$	$3\,\bar{3}$	$\bar{7}\,3$	$1\,\bar{3}$	$5\,3$	$\bar{1}\,\bar{3}$	$3\,\bar{3}$
$\bar{7}\,5$	$5\,5$	$\bar{3}\,5$	$7\,5$	$\bar{1}\,5$	$\bar{5}\,5$	$1\,\bar{5}$	$3\,\bar{5}$
$\bar{7}\,7$	$5\,7$	$\bar{3}\,\bar{7}$	$7\,7$	$\bar{1}\,\bar{7}$	$\bar{5}\,7$	$1\,\bar{7}$	$3\,\bar{7}$

and another with each pair of figures reversed, viz. $\bar{1}\,7$, $1\,5$, &c., as already seen. By substituting 1, 2, 3, ... for $\bar{7}\,7$, $\bar{7}\,5$, $\bar{7}\,3$, ... we obtain magic squares formed of the natural series from 1 to 64.

If in a square of side 8, *rstu* represent any four numerically unequal numbers of the series $\bar{4}$, $\bar{3}$, ... 3, 4, and consequently $\bar{r}$ $\bar{s}$ $\bar{t}$ $\bar{u}$ the other four, and it be assumed that in every such set as *Irs* there are like signs at the four points r, s, $\bar{r}$, $\bar{s}$, it is impossible to combine such sets as *Irs*, *IIrt*. For in *Irs* we necessarily have like signs at t, u, $\bar{t}$, $\bar{u}$, and consequently unlike signs at r, $\bar{t}$. So that *IIrt* has like signs at r, s as well as at r, t, $\bar{r}$, $\bar{t}$, and consequently like signs at 5 points, viz. r, s, $\bar{r}$, t, $\bar{t}$ which is impossible. The same reasoning applies to *Irs* with *II* $\bar{r}$ t or *II* r $\bar{t}$ or *II* $\bar{r}$ $\bar{t}$.

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In fact if we assume that any such form as *IIrt* has like signs at $r, t, \bar{r}, \bar{t}$, the four forms $II\,r\,t$, $II\,\bar{r}\,t$, $II\,r\,\bar{t}$, $II\,\bar{r}\,\bar{t}$ must have exactly the same character, and differ only as to their positions in the square. In consequence of this restriction, the only possible combination of two such squares is $I\,r\,s, t\,u, \bar{r}\,\bar{s}, \bar{t}\,\bar{u}$ with $II\,r\,s, t\,u, \bar{r}\,\bar{s}, \bar{t}\,\bar{u}$ or $II\,r\,\bar{s}, t\,\bar{u}, \bar{r}\,s, \bar{t}\,u$, which are all three the same. *e.g.*

I. and II.

	$1\,\bar{3}$	$2\,4$	$\bar{1}\,3$	$\bar{2}\,\bar{4}$
4	$\bar{a}$	b	$\bar{c}$	d
3	a	$\bar{b}$	c	$\bar{d}$
2	$\bar{a}$	b	$\bar{c}$	d
1	a	$\bar{b}$	c	$\bar{d}$
$\bar{1}$	a	$\bar{b}$	c	$\bar{d}$
$\bar{2}$	$\bar{a}$	b	$\bar{c}$	d
$\bar{3}$	a	$\bar{b}$	c	$\bar{d}$
$\bar{4}$	$\bar{a}$	b	$\bar{c}$	d

whence may be derived such squares as

$1\,\bar{3}$	$\bar{5}\,3$	$\bar{1}\,\bar{3}$	$\bar{7}\,3$	$5\,3$	$3\,\bar{3}$	$7\,3$	$\bar{3}\,3$
$\bar{1}\,5$	$5\,\bar{5}$	$1\,5$	$7\,\bar{5}$	$\bar{5}\,\bar{5}$	$\bar{3}\,5$	$\bar{7}\,\bar{5}$	$3\,5$
$1\,3$	$\bar{5}\,\bar{3}$	$\bar{1}\,3$	$\bar{7}\,\bar{3}$	$5\,\bar{3}$	$3\,3$	$7\,\bar{3}$	$\bar{3}\,\bar{3}$
$\bar{1}\,\bar{1}$	$5\,1$	$1\,\bar{1}$	$7\,1$	$\bar{5}\,1$	$\bar{3}\,\bar{1}$	$\bar{7}\,1$	$3\,\bar{1}$
$\bar{1}\,\bar{5}$	$5\,5$	$1\,\bar{5}$	$7\,5$	$\bar{5}\,5$	$\bar{3}\,\bar{5}$	$\bar{7}\,5$	$3\,\bar{5}$
$1\,\bar{7}$	$\bar{5}\,7$	$\bar{1}\,\bar{7}$	$\bar{7}\,7$	$5\,7$	$3\,\bar{7}$	$7\,7$	$\bar{3}\,\bar{7}$
$\bar{1}\,1$	$5\,\bar{1}$	$1\,1$	$7\,\bar{1}$	$\bar{5}\,\bar{1}$	$\bar{3}\,1$	$\bar{7}\,\bar{1}$	$3\,1$
$1\,7$	$\bar{5}\,\bar{7}$	$\bar{1}\,7$	$\bar{7}\,\bar{7}$	$5\,\bar{7}$	$3\,7$	$7\,\bar{7}$	$\bar{3}\,7$

and its reverse $\bar{3}\,1$, $3\,\bar{5}$, &c., which may be reduced to the natural scale, as before.

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ON CONCYCLIC CONICOIDS.

By HENRY M. JEFFERY, M.A.

1. THE following memoir may be considered an exercise in Duality, founded on that Chapter(VIII) of Dr. Salmon's *Geometry of Three Dimensions*, which treats of Confocal Conicoids.* It may be a useful preliminary to the study of the dual curvature of surfaces. The reader is referred to some of the demonstrations, which appear in that chapter; it being premised that Boothian and Cartesian coordinates are exchanged, as occasion may require.†

On Umbilics of Ellipsoids and their Duals.

2. The parallel circular sections of an ellipsoid terminate in four umbilics, at which the radius of curvature of all normal sections is the same, viz. $\frac{b^2}{ac}$.

By du-umbilics of a surface may be denoted points in which the curvature is that of the osculating conicoid of revolution, whose focus is the origin, for every plane section through the radius-vector. In the ellipsoid there are four real du-umbilics, at which points the asymptotes to the focal hyperbola meet it, and the semi-latus rectum of each focal conicoid of revolution, which measures the dual curvature, is also $\frac{b^2}{ac}$. For in a conic, $l=\frac{a^2c^2}{r^3}=\frac{b^2}{ac}$ (*Journal*, Vol. x., p. 323, on the Dual Curvature of Curves).

(Strictly speaking, the dual of an umbilic is the du-umbilical tangent plane).

* The theory of confocal ellipsoids was first considered by Maclaurin in treating of the attraction of ellipsoids, and finally established by M. Chasles (Briot et Bouquet, *Géométrie Analytique*, p. 432). Dr. Salmon has also briefly sketched out the reciprocal doctrine of concyclic conicoids (p. 157).

† The duals of Cartesian coordinates have been designated Boothian, as Dr. Booth introduced them into notice here (*Tangential Coordinates*, Dublin, 1840). But they were first conceived by MM. Chasles and Plücker, to establish analytically the principle of duality, which had been previously recognised by the method of reciprocal polars. (Briot et Bouquet, p 388).

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For the sake of distinctness, the following properties of the ellipse, and the confocal and concyclic hyperbolæ (the former being the focal conic) are considered in one plane (xz). See fig. 31.

Their equations are

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \text{ or } a^2\xi^2 + c^2\zeta^2 = 1 \ldots\ldots\ldots (A),$$

and

$$\frac{x^2}{a^2 - b^2} - \frac{z^2}{b^2 - c^2} = 1 \ldots\ldots\ldots (B),$$

$$-\frac{\xi^2}{\frac{1}{b^2} - \frac{1}{a^2}} + \frac{\zeta^2}{\frac{1}{c^2} - \frac{1}{b^2}} = 1 \ldots\ldots\ldots (C).$$

The conics (A) and (B) intersect in umbilics, the common tangents to (A) and (C) touch (A) in du-umbilics. (The conic (C) is the boundary between concyclic hyperboloids of one and two sheets).

3. (1) The asymptotes of the confocal hyperbola pass through the du-umbilics. (They also intersect the auxiliary circle $\{x^2 + z^2 = a^2\}$ in points corresponding to the umbilics. In other words, the umbilics have for their representative direction* the asymptotes of the confocal hyperbola).

(1) The asymptotes of the concyclic hyperbola are parallel to the tangents at umbilics. (Tangents to the auxiliary circle, which are parallel to these asymptotes touch it in points tangentially corresponding to the du-umbilics).

If the equation to the extremities of the asymptotes of the concyclic hyperbola be combined with that to the ellipse,

$$\frac{a^2\xi^2}{a^2 - b^2} = \frac{c^2\zeta^2}{b^2 - c^2} = \frac{1}{a^2 - c^2},$$

which determine the tangents at an umbilic.

If they are combined with those of the auxiliary circle,

$$\frac{a^2\xi^2}{a^2 - b^2} = \frac{c^2\zeta^2}{b^2 - c^2} = \frac{c^2}{b^2(a^2 - c^2)}.$$

(2) Tangents to the ellipse which are parallel to the asymptotes of the confocal

(2) The asymptotes of the concyclic hyperbola, lie in the planes of circular section.

* In general the direction (α, β, γ) is said to be the representative direction of the point $a\cos\alpha$, $b\cos\beta$, $c\cos\gamma$.

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hyperbola, lie at the distance of the semi-minor axis from the centre.

If the equation to the extremities of the asymptotes of the confocal hyperbola be combined with that to the ellipse,

$$\frac{\xi^2}{b^2-c^2} = \frac{\zeta^2}{a^2-b^2} = \frac{1}{p^2(a^2-c^2)} = \frac{1}{b^2(a^2-c^2)}.$$

(3) The ordinate through a du-umbilic meets a normal to a plane of circular section in the auxiliary circle. In other words, a du-umbilic has for its representative direction the normal to a plane of circular sections. (This property may be taken as a convenient constructive definition of the du-umbilic).

(3) The tangent drawn to the auxiliary circle from the point, in which the tangent to an umbilic meets the ξ line of reference, is perpendicular to the asymptote of the confocal hyperbola.

(4) If a circle be described concentric with the ellipse and with radius $\left(\frac{ac}{b}\right)$, it will intersect it in the du-umbilics.

(4) The common tangents to this circle of radius $\left(\frac{ac}{b}\right)$, and to the ellipse, touch the latter in umbilics, and the circle at its intersection with the perpendiculars to the planes of circular section.

(5) If a circle be described with radius $\left\{\frac{ac}{\sqrt{(a^2-b^2+c^2)}}\right\}$, it will intersect the ellipse, where lines perpendicular to the radii-vectores of the du-umbilics meet it.

(5) The common tangents of this circle and the ellipse touch the latter in the extremities of diameters conjugate with the radii-vectores of umbilics.

(6) There are two planes of circular section through the centre of a central conicoid.

(6) Two right cylinders can be circumscribed about a central conicoid, the edges of the cylinders being parallel to the asymptotes of the focal hyperbola. Salmon, p. 133.

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(7) The poles of all circular sections lie in the radii-vectores through the umbilics.

(7) The shadows cast by points in the radii-vectores drawn through the du-umbilics on the planes perpendicular to them are conics with a common focus in the centre of the ellipsoid.

Let the new lines of reference be inclined to the old at angle θ, so that

$$\xi = \xi' \cos\theta - \zeta' \sin\theta, \quad \eta = \xi' \sin\theta + \zeta' \cos\theta,$$

where $$\cos^2\theta = \frac{a^2 - b^2}{a^2 - c^2}, \quad \sin^2\theta = \frac{b^2 - c^2}{a^2 - c^2},$$

determining the du-umbilics.

The transformed equation to the ellipsoid

$$(a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 = 1)$$

becomes $$\xi^2 \frac{a^2c^2}{b^2} + \eta^2 b^2 + \left[b\zeta + \frac{\xi}{b} \sqrt{\{(a^2 - b^2)(b^2 - c^2)\}} \right]^2 = 1.$$

By assigning various values to ξ, the property may be established. The limiting cases of these conics are circles noticed above, and points, when $\xi = \frac{b}{ac}$ or the luminous point is a du-umbilic.

(8) Any two of the cones, which determine the shadows in two different systems, may be circumscribed about the same conicoid of revolution, whose focus is the centre of the ellipsoid.

The proof is the same as for the familiar dual theorem.

With the axes as lines of reference, combine the equations to two points in the radii-vectores through du-umbilics, with the equation to the ellipsoid.

$$\{\xi \sqrt{(a^2 - b^2)} + \zeta \sqrt{(b^2 - c^2)} + h\} \{\xi \sqrt{(a^2 - b^2)} - \zeta \sqrt{(b^2 - c^2)} + k\}$$
$$- (a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 - 1) = 0.$$

The resulting equation denotes a conicoid of revolution

$$\xi^2 + \eta^2 + \zeta^2 + A\xi + B\zeta + C = 0.$$

(9) The radius-vector of umbilic is equal to a

(9) The perpendicular on the tangent at a du-umbilic is

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perpendicular through a dumbilic to a tangent.

equal to the radius-vector, which is perpendicular to a plane of circular section.

On Concyclic Conicoids.

4. Concyclic conicoids have been also designated *codirective*, *i.e.* as having a common director cone. See *Journal*, Vol. IX., pp. 314—5, where the theory of the director-cone, the dual of the focal conic, is detailed. If the centre of reciprocation is, as in the following pages, the centre of the conicoid, the director-cone becomes the cylinder, whose principal section is a concyclic conic.

5. (1) Two conicoids are confocal if the differences of the squares of the corresponding axes are the same for both, or if the difference of the squares of the perpendiculars drawn on two parallel tangent planes is constant.

(1) Two conicoids are concyclic if the differences of the squares of the reciprocals of the corresponding axes are the same for both, or if the difference of the squares of the reciprocals of collinear radii-vectores is constant.

Thus the equation to a system of concyclic conicoids may be written indifferently

$$\frac{\xi^2}{\frac{1}{a^2}+\lambda}+\frac{\eta^2}{\frac{1}{b^2}+\lambda}+\frac{\zeta^2}{\frac{1}{c^2}+\lambda}=1,$$

$$x^2\left(\frac{1}{a^2}+\lambda\right)+y^2\left(\frac{1}{b^2}+\lambda\right)+z^2\left(\frac{1}{c^2}+\lambda\right)=1.$$

(2) Through a given point three conicoids can be drawn confocal to a given ellipsoid, viz. (1) an ellipsoid, (2) a hyperboloid of one sheet, and (3) another of two sheets. Salmon, p. 131.

(2) Three conicoids can be drawn touching a given plane concyclic to a given ellipsoid, viz. (1) an ellipsoid, (2) a hyperboloid of one sheet, and (3) another of two sheets.

(3) Two confocal conics or conicoids of different species cut each other at right angles. See fig. 31.

(3) If two concyclic conics reconicoids of different species have a common tangent, the points of contact are quadrantal, *i.e.* subtend a right angle at the origin. See fig. 31.

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(4) Where three confocals intersect the tangent plane to any one contains the normals to the other two.

(4) If three concyclics have a common tangent plane, the du-normals of each point of contact pass through the other two. Salmon, p. 97.

Def. By du-normal is meant that line in the tangent plane which is quadrantal with the point. Its equation is thus expressed by Boothian coordinates.

$$\frac{\xi' - \xi}{\frac{d\phi}{d\xi}} = \frac{\eta' - \eta}{\frac{d\phi}{d\eta}} = \frac{\zeta' - \zeta}{\frac{d\phi}{d\zeta}}.$$

6. (1) The axes of any tangent cone to a conicoid are the normals to three confocals, which can be drawn through the vertex of the cone.*

(1) The axes of the cone, whose vertex is the centre of the ellipsoid and dirigent conic a plane section of the ellipsoid, pass through the points of contact with that plane of three concyclic conicoids.

Two proofs of this theorem of M. Chasles are given, (1) by Briot and Bouquet, *Geom. Anal.*, p. 429; (2) by Salmon, p. 139. To these a third solution (3) is here added in Boothian coordinates.

(2) If the vertex of the enveloping cone be the new origin, and the three normals to the confocals through the vertex the new rectangular axes, the transformed equations to the cone and ellipsoid are (p. 140) shewn to be

$$\frac{x^2}{\mu_1} + \frac{y^2}{\mu_2} + \frac{z^2}{\mu_3} = 0,$$

$$\frac{x^2}{\mu_1} + \frac{y^2}{\mu_2} + \frac{z^2}{\mu_3} = S\left(\frac{p_1 x}{\mu_1} + \frac{p_2 y}{\mu_2} + \frac{p_3 z}{\mu_3} + 1\right)^2,$$

where $a^2+\mu_1$, $a^2+\mu_2$, $a^2+\mu_3$ denote the squares of the principal axes of the confocals, and S denotes $\frac{\mu_1\mu_2\mu_3}{a^2b^2c^2}$.

* The axes form with the lines through their intersections with the plane of contact, a tetrahedron which is self-conjugate with respect to the conicoid.

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(3) Proof of the dual theorem, which again, being dually interpreted, furnishes a third solution of Chasles' theorem.

Let a plane $(x\xi + y\eta + z\zeta = 1)$ cut the ellipsoid

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right),$$

the equation to the cone through the dirigent conic of intersection is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (x\xi + y\eta + z\zeta)^2.$$

Take as new axes the lines (r_1, r_2, r_3) joining the centre with the points of contact of the three concyclic conicoids, so that the old coordinates expressed in terms of the new have the form

$$\frac{\xi x}{r_1\left(\frac{1}{a^2} + \lambda_1\right)} + \frac{\xi y}{r_2\left(\frac{1}{a^2} + \lambda_2\right)} + \frac{\xi z}{r_3\left(\frac{1}{a^2} + \lambda_3\right)}.$$

After transformation, $x\xi + y\eta + z\zeta$ becomes

$$\frac{x}{r_1} + \frac{y}{r_2} + \frac{z}{r_3}, \text{ and } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \text{ becomes}$$

$$\frac{x^2}{r_1^2}\left\{\frac{\xi^2}{a^2\left(\frac{1}{a^2} + \lambda_1\right)^2} + \frac{\eta^2}{b^2\left(\frac{1}{b^2} + \lambda_1\right)^2} + \frac{\zeta^2}{c^2\left(\frac{1}{c^2} + \lambda_1\right)^2}\right\}$$

$$+ \frac{2xy}{r_1 r_2}\left\{\frac{\xi^2}{a^2\left(\frac{1}{a^2} + \lambda_1\right)\left(\frac{1}{a^2} + \lambda_2\right)} + \frac{\eta^2}{b^2\left(\frac{1}{b^2} + \lambda_1\right)\left(\frac{1}{b^2} + \lambda_2\right)}\right.$$

$$\left. + \frac{\zeta^2}{c^2\left(\frac{1}{c^2} + \lambda_1\right)\left(\frac{1}{c^2} + \lambda_2\right)}\right\} + \ldots\ldots$$

or $$\left(\frac{x}{r_1} + \frac{y}{r_2} + \frac{z}{r_3}\right)^2 - (\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2).$$

The transformed equation to the ellipsoid assumes the remarkable form

$$\left(\frac{x}{r_1} + \frac{y}{r_2} + \frac{z}{r_3}\right)^2 - (\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2) = 1 \ldots\ldots (A).$$

The dual theorem is proved by the form of the equation to the cone

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0.$$

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The equation to the concyclic

$$x^2\left(\frac{1}{a^2}+\lambda_1\right)+y^2\left(\frac{1}{b^2}+\lambda_1\right)+z^2\left(\frac{1}{c^2}+\lambda_2\right)=1$$

becomes, when transformed,

$$\left(\frac{x}{r_1}+\frac{y}{r_2}+\frac{z}{r_3}\right)^2+(\lambda_1-\lambda_2)\,y^2+(\lambda_1-\lambda_3)\,z^2=1,$$

(since $x^2+y^2+z^2$ is unaltered).

(4) Dually viewed, the Boothian equation to the ellipsoid, when the new lines of reference are parallel to the normals of the confocals through a given point (x', y', z'), may be written

$$(p_1\xi+p_2\eta+p_3\zeta)^2-(\mu_1\xi^2+\mu_2\eta^2+\mu_3\zeta^2)=1\ldots(B).$$

(5) The Cartesian equation to the ellipsoid referred to the new axes, but with origin unaltered, may be written in the form

$$\frac{x^2}{\mu_1}+\frac{y^2}{\mu_2}+\frac{z^2}{\mu_3}+1=S\left\{\frac{xp_1}{\mu_1}+\frac{yp_2}{\mu_2}+\frac{zp_3}{\mu_3}\right\}^2\ldots.(C).$$

The enveloping cone is thus denoted in this system:

$$\frac{x^2}{\mu_1}+\frac{y^2}{\mu_2}+\frac{z^2}{\mu_3}+1=2\left\{\frac{xp_1}{\mu_1}+\frac{yp_2}{\mu_2}+\frac{zp_3}{\mu_3}\right\}-\frac{1}{S},$$

with dual Boothian forms for the ellipsoid and dirigent conic referred to certain radii-vectores above described as lines of reference.

The forms (A), (B), (C), and (D), the Boothian dual of (C), are important transformations of the equation to a central conicoid.

7. It is next proposed to consider the dual interpretations of certain linear equations, in the transformed system of coordinate axes

$$\frac{p_1x}{\mu_1}+\frac{p_2y}{\mu_2}+\frac{p_3z}{\mu_3}=\frac{1}{S}\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{p_1x}{\mu_1}+\frac{p_2y}{\mu_2}+\frac{p_3z}{\mu_3}=\pm\frac{1}{\sqrt{S}}\ldots\ldots\ldots\ldots(2),$$

$$\frac{p_1x}{\mu_1}+\frac{p_2y}{\mu_2}+\frac{p_3z}{\mu_3}=1+\frac{1}{S}\ldots\ldots\ldots\ldots\ldots(3),$$

$$\frac{p_1x}{\mu_1}+\frac{p_2y}{\mu_2}+\frac{p_3z}{\mu_3}=\frac{1}{2}\left(1+\frac{1}{S}\right)\ldots\ldots\ldots(4).$$

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They may be thus interpreted:

(1) The plane of contact of the cone enveloping the given ellipsoid.	(1) The vertex of the cone enveloping the given ellipsoid.
(2) The intersections with the ellipsoid of a cone asymptotic to the auxiliary hyperboloid (§ 8), whose vertex is the centre, equal and parallel to the enveloping cone. The distances of (1) and (2) from the centre are as $abc : \sqrt{(\mu_1\mu_2\mu_3)}$.	(2) The vertices of enveloping cones, which are equal and parallel to the cone through the dirigent conic, which has the origin for its vertex.
(3) The parallel plane through the vertex of the enveloping cone. $\left(\text{Hence } \frac{p_1^2}{\mu_1}+\frac{p_2^2}{\mu_2}+\frac{p_3^2}{\mu_3}=1+\frac{1}{S}\right)$	(3) The intersection of the line joining the points (1) and (2) with the given plane.
(4) The intersection of the two cones which is at half the distance of (3) from the centre.	(4) The vertex of the cone passing through the dirigent conic, which is equal and parallel to the former cone through the same conic.

8. An auxiliary hyperboloid of one sheet touches the given ellipsoid in a section parallel to the plane of contact, and belongs as a conjugate to a second system of confocals passing through the point of intersection of the original confocals, noticed below in § 11.

It is thus defined:

$$\frac{x^2}{\mu_1}+\frac{y^2}{\mu_2}+\frac{z^2}{\mu_3}+1=0, \text{ or } -\frac{x^2}{\rho_1^2}+\frac{y^2}{\rho_2^2}+\frac{z^2}{\rho_3^2}=1,$$

since μ_2, μ_3 are essentially negative (Salmon, p. 131), and μ_1 is positive for an external point. The hyperboloid is a surface of revolution, if the external point lie on a focal conic of the ellipsoid, see § 14.

9. It is desirable next to transpose the equations of the confocals through (x', y', z'), as in Salmon, p. 139.

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The equation $\frac{x^2}{a^2+\mu_1}+\frac{y^2}{b^2+\mu_1}+\frac{z^2}{c^2+\mu_1}=1$, becomes

$$p_1^4x^2\left\{\frac{x'^2}{(a^2+\mu_1)^3}+\frac{y'^2}{(b^2+\mu_1)^3}+\frac{z'^2}{(c^2+\mu_1)^3}\right\}-\frac{2xyp_2}{p_1(\mu_1-\mu_2)}-\frac{2xzp_3}{p_1(\mu_1-\mu_3)}+\frac{y^2}{\mu_1-\mu_2}+\frac{z^2}{\mu_1-\mu_3}=1.$$

Since the points (p_1, p_2, p_3) lies on the surface

$$p_1^4\left\{\frac{x'^2}{(a^2+\mu_1)^3}+\frac{y'^2}{(b^2+\mu_1)^3}+\frac{z'^2}{(c^2+\mu_1)^3}\right\}-\frac{p_2^2}{\mu_1-\mu_2}-\frac{p_3^2}{\mu_1-\mu_3}=1,$$

the equation may be written

$$x^2+\frac{(yp_1-xp_2)^2}{\mu_1-\mu_2}+\frac{(zp_1-xp_3)^2}{\mu_1-\mu_3}=p_1^2,$$

and similar equations for the other two confocals.

If the origin be transformed to the point of intersection (p_1, p_2, p_3), the section by a tangent plane is thus denoted:

$$\frac{y^2}{\mu_1-\mu_2}+\frac{z^2}{\mu_1-\mu_3}=0.$$

10. Hence the following theorems readily follow:

(1) The focal lines of a system of cones which have a common vertex and envelop confocal conicoids, are generating lines of the confocal hyperboloid which passes through the point.

(1) The circular sections of a system of cones whose vertex is the centre, and dirigent conics formed by the intersection of a fixed plane with concyclic conicoids, contain generating lines of the concyclic hyperboloid at the hyperbolic point of contact, where it is touched by the given plane.

(2) The sections of the confocals by planes through the centre parallel to the tangent planes, are the focal conics of an auxiliary hyperboloid. See § 12, (1).

(2) The principal sections of the cylinders enveloping concyclics, whose generating lines are parallel to the radii-vectores drawn to the points of contact, are the cyclic conics of an auxiliary hyperboloid.

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(3) The axes of such a central section are parallel to the normals to the two confocals.

(3) The axes of the principal sections of such circumscribing cylinders pass through the points of contact of the two concyclics.

(4) The lengths of these axes are those of the confocal conics of an auxiliary hyperboloid.

(4) The lengths of the axes are those of the concyclic conics of an auxiliary hyperboloid.

For separate proofs, see Salmon, pp. 134, 135.

11. If three additional confocals be described whose axes are those of the original confocals, but in cross-order, and measured in the direction of the normals at the point of intersection, they will intersect in the same point, and their tangent planes will be parallel to the original principal planes.

If three additional concyclics have the axes of the original concyclics, but in cross-order, and measured along the radii-vectores to the quadrantal points, they have the same common plane of contact, the quadrantal points of contact lying in the primitive principal axes.

The equations to the new confocals are

$$\frac{x^2}{a^2+\mu_1}+\frac{y^2}{a^2+\mu_2}+\frac{z^2}{a^2+\mu_3}=1,$$

$$\frac{x^2}{b^2+\mu_1}+\frac{y^2}{b^2+\mu_2}+\frac{z^2}{b^2+\mu_3}=1,$$

$$\frac{x^2}{c^2+\mu_1}+\frac{y^2}{c^2+\mu_2}+\frac{z^2}{c^2+\mu_3}=1,$$

which determine the same point of intersection as for the original confocal system.* When transformed to the original principal axes, they assume the form

$$x^2+\frac{(yX-xY)^2}{a^2-b^2}+\frac{(zX-xZ)^2}{a^2-c^2}=1.$$

* These surfaces are confocal with the hyperboloid of two sheets $\left(\frac{x^2}{\mu_1}+\frac{y^2}{\mu_2}+\frac{z^2}{\mu_3}=1\right)$, which by a change of axes may be shewn, as in § 8, to osculate the imaginary conjugate $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+1=0\right)$ of the given ellipsoid.

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Similarly, the equations of the new concyclics have this form

$$(x\xi + y\eta + z\zeta)^2 - \left(\frac{1}{b^2} - \frac{1}{a^2}\right)y^2 - \left(\frac{1}{c^2} - \frac{1}{a^2}\right)z^2 = 1,$$

if (ξ, η, ζ) denote the common plane.

The Boothian duals of these equations will point out the corresponding properties.

12. (1) The sections of one set of concurrent confocals made by the principal planes of the other set, are the conjugates of the focal conics of the latter.

(1) The sections of one set of co-tangential concylics made by a plane through the centre parallel to the common tangent plane are conics, which, when orthogonally projected on the principal planes of the other set of concyclics, are the conjugates of its concyclic conics.

(2) The cylinders, circumscribed along these curves of intersection, have their generating lines parallel to the radii-vectores at the points of concurrence.

(2) The cylinders, which are circumscribed about these curves of intersection, have their generating lines parallel to the lines of reference.

Def. As *focal cones* are those, of which the focal conics are dirigent and the vertex indefinite, so *cyclic cones* are those, whose vertex is the origin, and whose dirigent conics are the intersections of any plane with the cylinders passing orthographically through the cyclic conics.

(3) The focal cones of one set of concurrent confocals are equal and parallel to the asymptotic cones of the other set of concurrent confocals.

(If the centre of one set of confocals were the point of intersection of the other set, the focal cones of one would be the asymptotic cones of the other).

(3) The cyclic cones of one set of concyclics are the asymptotic cones of the other set of concyclics, which are co-tangential with the former set.

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13. The preceding theorems are repeated for plane conics.

If there be two sets of concurrent confocal conics, each ellipse passes through the foci of the other; the tangents at the foci are parallel to the radii-vectores at the points of concurrence, and the asymptotes of one hyperbola are parallel to the lines which join the common point to the foci of the other. See fig. 32. The equations to the two sets of confocal conics are

$$\left.\begin{aligned} \frac{x^2}{a^2} + \frac{z^2}{c^2} &= 1 \\ \frac{x^2}{a^2-b^2} - \frac{z^2}{b^2-c^2} &= 1 \end{aligned}\right\} \dots (1),$$

$$\left.\begin{aligned} \frac{x^2}{a^2} + \frac{z^2}{a^2-b^2} &= 1 \\ \frac{x^2}{c^2} + \frac{z^2}{b^2-c^2} &= 1 \end{aligned}\right\} \dots (2).$$

The equations (2) (whose axes of figure are parallel to the tangents to (1)), when transformed to the axes of (1), become

$$z^2 + \frac{(xz_1 - zx_1)^2}{a^2-c^2} = z_1^2,$$

$$x^2 - \frac{(zx_1 - xz_1)^2}{a^2-c^2} = x_1^2.$$

To these may be added the dual forms:

$$(x_1\xi + z_1\zeta)^2 + (a^2-c^2)\,\zeta^2 = 1,$$

$$(x_1\xi + z_1\zeta)^2 - (a^2-c^2)\,\xi^2 = 1.$$

If there be two sets of cotangential concyclic conics, the points of contact of one set with the common tangent line lie in the principal axes of the other set; lines parallel to the common tangent meet the ellipses in points at which the tangents are parallel to lines of reference; and the asymptotes of the one hyperbola intersect the common line in points, which are orthogonally projected in fixed points of the transverse axis of the other hyperbola. See fig. 33. The two sets of concyclics are thus denoted, the axes of the one being radii-vectores to the points of contact of the other in the common tangent line (ξ, ζ),

$$\left.\begin{aligned} \frac{x^2}{a^2} + \frac{z^2}{c^2} &= 1 \\ -\left(\frac{1}{b^2} - \frac{1}{a^2}\right)x^2 + \left(\frac{1}{c^2} - \frac{1}{b^2}\right)z^2 &= 1 \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{x^2}{a^2} - \left(\frac{1}{b^2} - \frac{1}{a^2}\right)z^2 &= 1 \\ \frac{x^2}{c^2} + \left(\frac{1}{c^2} - \frac{1}{b^2}\right)z^2 &= 1 \end{aligned}\right\} \dots (2).$$

These equations (2), when transformed to the axes of (1), become

$$(x\xi + z\zeta)^2 + \frac{a^2-c^2}{a^2c^2}x^2 = 1,$$

$$(x\xi + z\zeta)^2 - \frac{a^2-c^2}{a^2c^2}z^2 = 1.$$

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To these may be added the dual forms:

$$\xi^2 + \frac{(\zeta\xi_1 - \xi\zeta_1)^2}{\frac{1}{c^2} - \frac{1}{a^2}} = \xi_1^2,$$

$$\zeta^2 - \frac{(\zeta\xi_1 - \xi\zeta_1)^2}{\frac{1}{c^2} - \frac{1}{a^2}} = \zeta_1^2.$$

14. The locus of the vertices of right cones, which can envelope an ellipsoid, is one of the focal conics.

If plane sections of an ellipsoid (drawn parallel to the principal planes) form the dirigent conics of right cones with their vertics in the centre, these planes envelope the concyclic conics.

The dual theorem may be established, as in Salmon, p. 149, from examining the general equation of such a cone (§ 6, (3)),

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0.$$

The following is a direct proof.

It is required to find the conditions that the following cone be right:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (y\eta + z\zeta)^2.$$

Change the axes, to avoid the term involving (yz), so that the new axis of y or z may be an axis of the cone,

$$\frac{2x^2}{a^2} + (y^2 + z^2)\left(\frac{1}{b^2} + \frac{1}{c^2} - \eta^2 - \zeta^2\right)$$

$$+ (y^2 - z^2)\sqrt{\left\{\left(\frac{1}{b^2} - \frac{1}{c^2} - \eta^2 + \zeta^2\right)^2 + 4\eta^2\zeta^2\right\}} = 0.$$

If the coefficients of x^2 and y^2 or z^2 are identified,

$$\left(\frac{2}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} + \eta^2 + \zeta^2\right)^2 = \left(\frac{1}{b^2} - \frac{1}{c^2} - \eta^2 + \zeta^2\right)^2 + 4\eta^2\zeta^2,$$

that is,

$$\frac{\eta^2}{\frac{1}{b^2} - \frac{1}{a^2}} + \frac{\zeta^2}{\frac{1}{c^2} - \frac{1}{a^2}} = 1.$$

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15. If a spherical ellipse be orthogonally projected on either of two principal planes of the ellipsoid, on which it lies, the section of the cylinder thus formed by the plane of circular section is a circle.

The orthographic projection of the developable formed by the tangent planes common to a sphere and ellipsoid (which touch the former in a spherical ellipse) on either of two principal planes is a conic, which conic, if projected on a principal section of the right cylinder, which is circumscribed about the ellipsoid, is a circle.

For the equation to the trace of the developable on a principal plane is

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=\frac{r^2}{r^2-c^2}.$$

That to its projection is

$$x^2+y^2=(b^2-c^2)\frac{r^2}{r^2-c^2},$$

if the conic be projected on the plane thus denoted;

$$x\sqrt{(a^2-b^2)}\pm z\sqrt{(b^2-c^2)}=0.$$

16. Two confocal conicoids can be drawn to touch a given line, and the tangent planes drawn at the points of contact are at right angles to each other.

Two concyclic conicoids can be drawn to touch a given line, and the points of contact are quadrantal.

(1) Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ be the two fixed points: and $\xi(a^2+\mu)$, $\eta(b^2+\mu)$, $\zeta(c^2+\mu)$ denote an assumed point of contact on the confocal of a given conicoid.

Since they are collinear,

$$\frac{\xi(a^2+\mu)-\alpha_1}{\alpha_2-\alpha_1}=\frac{\eta(b^2+\mu)-\beta_1}{\beta_2-\beta_1}=\frac{\zeta(c^2+\mu)-\gamma_1}{\gamma_2-\gamma_1},$$

The eliminant of these and the two equations

$$\alpha_1\xi+\beta_1\eta+\gamma_1\zeta=1,\quad \alpha_2\xi+\beta_2\eta+\gamma_2\zeta=1,$$

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line $\frac{\xi}{\alpha_1-\alpha_2}=\frac{\eta}{\beta_1-\beta_2}=\frac{\zeta}{\gamma_1-\gamma_2}$ lies on the hyperboloid. In this line, planes perpendicular to the line joining the fixed points meet infinity.

One of the generators is the polar line of the given line with respect to any confocal, as may be shewn by the equation

$$\frac{1}{\alpha_1-\alpha_2}\left(\xi-\frac{\alpha_1}{a^2+\mu}\right)=\frac{1}{\beta_1-\beta_2}\left(\eta-\frac{\beta_1}{b^2+\mu}\right)=\frac{1}{\gamma_1-\gamma_2}\left(\zeta-\frac{\gamma_1}{c^2+\mu}\right).$$

The equation to the surface may also be written

$$\eta\zeta(\alpha_1-\alpha_2)(b^2-c^2)+\zeta\xi(\beta_1-\beta_2)(c^2-a^2)+\xi\eta(\gamma_1-\gamma_2)(a^2-b^2)$$
$$=\xi(\beta_1\gamma_2-\beta_2\gamma_1)+\eta(\gamma_1\alpha_2-\gamma_2\alpha_1)+\zeta(\alpha_1\beta_2-\alpha_2\beta_1).$$

Hence, the pole of the plane at infinity with respect to the surface is also the quadrantal pole of the plane through the origin and the fixed points.

The orthographic projections on the principal planes are parabolas touching the axes.

If the given line be a normal at a confocal to a tangent (α, β, γ), the equation to the locus becomes

$$(\alpha\eta\zeta+\beta\gamma\xi)(b^2-c^2)+(\beta\zeta\xi+\gamma\alpha\eta)(c^2-a^2)$$
$$+(\gamma\xi\eta+\alpha\beta\zeta)(a^2-b^2)=0.$$

Since the discriminant is zero, it denotes a plane conic, viz. a parabola. The polar line of the fixed normal is perpendicular to it, and passes through each point of the parabola cylindrically.

In the dual theorem, if the du-normal to a concyclic be the intersecting line, the developable surface is a cone.

18. The locus of the points of contact of parallel planes, touching a system of confocal conicoids, is a hyperbola.

The envelop of tangent planes drawn at collinear points in a system of concyclic conicoids is a conic.

If $lx+my+nz=0$ denote a parallel plane to the system, the locus may be shewn to be defined by these two equations:

$$\frac{x}{l}(b^2-c^2)+\frac{y}{m}(c^2-a^2)+\frac{z}{n}(a^2-b^2)=0.$$

$$\frac{lx}{b^2-c^2}\left(\frac{y}{m}-\frac{z}{n}\right)+\frac{my}{c^2-a^2}\left(\frac{z}{n}-\frac{x}{l}\right)+\frac{nz}{a^2-b^2}\left(\frac{x}{l}-\frac{y}{m}\right)=1.$$

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19. *Def.* Corresponding planes, which touch concyclics of the same species, such as ellipsoids, pass through straight lines (du-normals of the points of contact), which touch two concyclic hyperboloids of different species. These planes touch the developable formed by tangent planes common to the hyperboloids.

The intercepts of corresponding planes on the axes are proportional to those axes.

Their Boothian coordinates are thus related:

$$\frac{\xi}{\xi'} = \frac{a'}{a}, \quad \frac{\eta}{\eta'} = \frac{b'}{b}, \quad \frac{\xi}{\xi'} = \frac{c'}{c}.$$

The difference of the reciprocals of the squares of the perpendiculars from the centre on two corresponding planes is constant, and equal to the difference of the reciprocals of the squares of the axes.

(The dual of Ivory's theorem). If OM, ON' denote the perpendiculars from the centre on the tangent planes to two concyclic conicoids, and OM', ON, those on the corresponding planes,

$$\frac{MN'}{OM.ON'} = \frac{M'N}{OM'.ON}.$$

20. Confocal conicoids are not co-tangential.

Concyclic conicoids do not intersect each other.

Since concyclic cones, which are asymptotic to concyclic conicoids, only meet in the vertex, there is no direct dual of MacCullagh's theorem on the intercept of the chord common to two focal conics.

21. The concluding theorems are connected with dual curvature.

(1) If a developable be formed by two concyclics, and radii-vectores be drawn to the points of contact of any common tangent line, the intercept (P) of either radius-vector by a tangent plane parallel to the other radius is constant.

(2) If a developable be formed by two concyclics, and r be the radius-vector drawn to a point in one of them, and p the perpendicular on a tangent plane, which is parallel to the central plane through the two above-named points of contact, rp is constant.

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Since r, p, P form a conjugate system,

$$rp = \frac{a_1 b_1 c_1}{P} = a_1 b_1 c_1 \sqrt{(\lambda_1 - \lambda_2)}.$$

The dual theorem is important in the theory of geodesic lines on conicoids.

(3) The polar of a given point with respect to a system of concyclics passes through the du-normal of that concyclic on which the point lies.

The polar of the point (f, g, h) with respect to a system of concyclics

$$\left\{x^2\left(\frac{1}{a^2}+\lambda\right)+y^2\left(\frac{1}{b^2}+\lambda\right)+z^2\left(\frac{1}{c^2}+\lambda\right)=1\right\}$$

passes through the intersection of the two planes

$$\left\{\frac{fx}{a^2}+\frac{gy}{b^2}+\frac{hz}{c^2}=1,\ \ fx+gy+hz=0\right\},$$

which determine the du-normal.

(4) If three concyclics (A), (B), (C) are co-tangential, the directrices of the two focal conicoids of principal curvature of (A) for the common tangent plane are the polar planes of the point of contact with (A) with respect to (B) and (C). This is the dual of Dr. Salmon's theorem (p. 161).

(5) The points of contact of the common tangent plane with (B) and (C) are the intersections of consecutive du-tangent lines (or intersections of consecutive tangent planes) to the dual curves of curvature, which pass through the point of contact with (A).

These dual curves of curvature may be thus defined. The vector-plane through the tangent to such a curve is perpendicular to the vector-plane through the du-tangent.

The locus of the points of contact with (B) and (C) is the dual of the surface of centres of a conicoid, and may be called its du-centro-surface.

Cheltenham, *Feb.* 4, 1871.

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ON THE ENVELOPE OF A CERTAIN QUADRIC SURFACE.

By Prof. CAYLEY.

To find the envelope of the quadric surface

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$

where the coefficients vary subject to the conditions

$$\begin{cases} a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 = 0, \\ \dfrac{p^2}{a} + \dfrac{q^2}{b} + \dfrac{r^2}{c} + \dfrac{s^2}{d} = 0, \end{cases}$$

$(\alpha, \beta, \gamma, \delta)$ and (p, q, r, s) being respectively constant.

We have in the usual manner

$$x^2 + \lambda\alpha^2 + \mu\frac{p^2}{a^2} = 0,$$

$$y^2 + \lambda\beta^2 + \mu\frac{q^2}{b^2} = 0,$$

$$z^2 + \lambda\gamma^2 + \mu\frac{r^2}{c^2} = 0,$$

$$w^2 + \lambda\delta^2 + \mu\frac{s^2}{d^2} = 0,$$

and thence $\alpha^2 = \dfrac{-\mu p^2}{x^2 + \lambda\alpha^2}$, &c., and substituting these values μ disappears and we have

$$p\sqrt{(x^2 + \lambda\alpha^2)} + q\sqrt{(y^2 + \lambda\beta^2)} + r\sqrt{(z^2 + \lambda\gamma^2)} + s\sqrt{(w^2 + \lambda\delta^2)} = 0,$$

$$\frac{\alpha^2 p}{\sqrt{(x^2 + \lambda\alpha^2)}} + \frac{\beta^2 q}{\sqrt{(y^2 + \lambda\beta^2)}} + \frac{\gamma^2 r}{\sqrt{(z^2 + \lambda\gamma^2)}} + \frac{\delta^2 s}{\sqrt{(w^2 + \lambda\delta^2)}} = 0,$$

from which λ is to be eliminated; the second equation is here the derived function of the first in regard to λ, so that rationalising the first equation, the result is, as will be shown, of the form $(*\chi\lambda, 1)^4 = 0$, and the result is obtained by equating to zero the discriminant of the quartic function.

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Denoting for shortness the first equation by

$$A + B + C + D = 0,$$

the rationalised form is

$$(A^4 + B^4 + C^4 + D^4 - 2A^2B^2 - 2A^2C^2 - 2A^2D^2 - 2B^2C^2 - 2B^2D^2 - 2C^2D^2)^2 - 64A^2B^2C^2D^2 = 0,$$

which is of the form

$$-(\mathfrak{A} + 2\mathfrak{B}\lambda + \mathfrak{C}\lambda^2)^2 + (a, b, c, d, e)(1, \lambda)^4 = 0,$$

where

$$\mathfrak{A} = p^4x^4 \ldots - 2p^2q^2x^2y^2 \ldots$$

$$\mathfrak{B} = p^4\alpha^2x^2 \ldots - p^2q^2(\alpha^2y^2 + \beta^2x^2) \ldots$$

$$\mathfrak{C} = p^4\alpha^4 \ldots - 2p^2q^2\alpha^2\beta^2 \ldots$$

$$a = 8.x^2y^2z^2w^2$$

$$4b = 8.\alpha^2y^2z^2w^2 + \ldots$$

$$6c = 8.\alpha^2\beta^2z^2w^2 + \ldots$$

$$4d = 8.\alpha^2\beta^2\gamma^2w^2 + \ldots$$

$$e = 8.\alpha^2\beta^2\gamma^2\delta^2.$$

Writing I', J' for the two invariants we find without difficulty

$$I' = I - \tfrac{1}{3}P + \Delta^2,$$

$$J' = J - Q + \tfrac{1}{3}\Delta P - \tfrac{8}{27}\Delta^3,$$

where

$$I = ae - 4bd + 3c^2,$$

$$J = ace - ad^2 - b^2e - c^3 + 2bcd,$$

$$\Delta = \mathfrak{A}\mathfrak{C} - \mathfrak{B}^2,$$

$$P = a\mathfrak{C}^2 - 4b\mathfrak{B}\mathfrak{C} + 2c(\mathfrak{A}\mathfrak{C} + 2\mathfrak{B}^2) - 4d\mathfrak{A}\mathfrak{B} + e\mathfrak{A}^2,$$

$$Q = (ce - d^2)\mathfrak{A}^2 + (ac + 2bd - 3c^2).\tfrac{1}{3}(\mathfrak{A}\mathfrak{C} + 2\mathfrak{B}^2) + (ac - b^2)\mathfrak{C}^2 - 2(ad - bc)\mathfrak{B}\mathfrak{C} - 2(bc - cd)\mathfrak{A}\mathfrak{B}.$$

The result thus is

$$(I - P + \tfrac{1}{3}\Delta^2)^3 - 27(J - Q + \tfrac{1}{3}\Delta P - \tfrac{8}{27}\Delta^3)^2 = 0,$$

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or what is the same thing, it is

$$(I-P)^3-27(J-Q)^2-9\Delta P(J-2Q)$$
$$+\Delta^2(4I^2-8IP+P^2)$$
$$+8\Delta^2(J-2Q)$$
$$+\Delta^4.\tfrac{1}{3}^6 I=0,$$

where the left-hand side is of the order 24 in (x, y, z, w). I apprehend that the order should be $=12$ only; for writing (x, y, z, w) in place of (x^2, y^2, z^2, w^2), the equations which connect (a, b, c, d) express that these quantities are the coordinates of a point on a plane cubic; and the problem is in fact that of finding the reciprocal of the plane cubic: this is a sextic cone, or restoring (x^2, y^2, z^2, w^2) instead of (x, y, z, w), we should have a surface of the order 12. I cannot explain how the reduction is effected.

TABLES OF THE BINARY CUBIC FORMS FOR THE NEGATIVE DETERMINANTS, $\equiv 0$ (MOD. 4), FROM -4 TO -400; AND $\equiv 1$ (MOD. 4), FROM -3 TO -99; AND FOR FIVE IRREGULAR NEGATIVE DETERMINANTS.

By Prof. CAYLEY.

THE theory of binary cubic forms for determinants, as well positive as negative, has been studied by M. Arndt in the memoir "Versuch einer Theorie der homogenen Functionen des dritten Grades mit zwei Variabeln," *Grunert's Archiv*, t. XVII. (1851, pp. 1–54) and in the later memoir, "Tabellarische Berechnung der reducirten binären cubischen Formen und Klassification derselben für alle negativen Determinanten $(-D)$ von $D=3$ bis $D\equiv 2000$," *ditto*, t. XXXI. (1858), pp. 335–445, he has given a very valuable Table of the forms for a Negative Determinant. It has appeared to me suitable to arrange this Table in the manner made use of for Quadratic Forms in my memoir "Tables des formes quadratiques binaires pour les déterminants negatifs $D\equiv -1$ jusqu'à $D\equiv -100$, pour les déterminants positifs non carrès depuis $D\equiv 2$ jusqu'à $D\equiv 99$, et pour les treize determinants

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negatifs du premier millier," *Crelle*, t. LX. (1862), pp. 357–372; and confining myself to the limits of the last mentioned tables I deduce from that of M. Arndt the three Tables which follow:

To explain the arrangement, I give in the first instance the following extract from M. Arndt's Table:

D.	*Reducirte Formen mit Charakteristik.*	*Klassen.*
3	(0,1,1,0) (1,0,−1,−1) (1,1,0,−1) (2,1,2) (2,1,2) (2,1,2)	(0, 1, 1, 0), (1, 0, −1, ±1)
4	(0, 1, 0, −1) (1, 0, −1, 0) (2, 0, 2) (2, 0, 2)	(0, −1, 0, 1)
7	(0, 1, 1, −1) (2, 1, 4)	(0, −1, −1, 1)
8	(0, 1, 0, −2) (2, 0, 4)	(0, −1, 0, 2)
11	(0, 1, 1, −2) (2, 1, 6)	(0, −1, −1, 2)
12	(0, 1, 0, −3) (2, 0, 6)	(0, −1, 0, 3)
15	(0, 1, 1, −3) (2, 1, 8)	(0, −1, −1, 3)
44	(0, 1, 0, −11) (1, −1, −2, 0) (2, 0, 22) (6, 2, 8)	(0,−1,0,11), (0,−2,±1,1)
112	(0,1,0,−28) (0,2,2,−2) (1,2,0,−4) (2,0,56) (8,4,16) (8,4,16) (1, −1, −3, −1) (8, 4, 16)	(0,−1,0,28), (0,2,2,−2), (1, ±1, −3, ±1)
144	(0, 1, 0, −36) (0, 2, 2, −3) (2, 0, 72) (8, 4, 20)	(0,−1,0,36), (0,−2,−2,3)
156	(0, 1, 0, −39) (1, −1, −3, 1) (2, 0, 78) (8, 2, 20)	(0,−1,0,39), (1,∓1,−3,±1)
216	(0,1,0,−54) (1,−2,−3,0) (2,0,−3,0) (2,0,108) (14,6,18) (12,0,18)	(0,−1,0,54), (0,∓3,0,±2), (0, ∓3, 2, ±1)

The first column contains the value of the determinant, the second column contains the *reduced* forms, omitting the

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contrary and *opposite* forms; viz. for the cubic form (a, b, c, d), the contrary form (equal, that is, properly equivalent to the given form) is $(-a, -b, -c, -d)$; and the opposite form (improperly equivalent to the given form) is $(a, -b, c, -d)$ or $(-a, b, -c, d)$; this second column contains also the *characteristic* of each cubic form, viz. the cubic form (a, b, c, d) has for its characteristic the quadratic form

$$\{2(b^2 - ac),\ bc - ad,\ 2(c^2 - bd)\},$$

(so that the cubic form and its characteristic have the same determinant

$$-D = (bc - ad)^2 - 4(b^2 - ac)(c^2 - bd),\ \equiv 1 \text{ or } 0 \pmod 4),$$

and a cubic form which corresponds to a reduced characteristic is itself a reduced form. The third column contains for each determinant the entire series of *unequal* cubic forms (that is of the forms whereof no two are properly equivalent to each other), the representatives of the *classes* for this determinant. M. Arndt has included in his table the non-primitive classes (for example Det. $= -112$, the form $(0, 2, 2, -2)$), for which the terms (a, b, c, d) have a common divisor μ, but as these are at once deducible from the classes which belong to the determinant $= -\frac{D}{\mu^4}$, its seems better to omit the non-primitive classes.

The two opposite forms included in a single expression by means of the sign $\pm$ have opposite characteristics which are for the most part unequal to each other, for instance

Det. -44; $(0, -2, 1, 1)$ has the characteristic $(6, -2, 8)$

$(0, 2, 1, -1)$ „ „ $(6, 2, 8)$,

where $(6, -2, 8)$, $(6, 2, 8)$ are unequal forms, but this is not always the case, for instance

Det. -112; $(1, -1, -3, -1)$ has the characteristic $(8, -4, 16)$

$(1, 1, -3, 1)$ „ „ $(8, 4, 16)$

where $(8, -4, 16) = (8, 4, 16)$, since each is an ambiguous form. Instead of the two unequal forms $(1, -1, -3, -1)$, $(1, 1, -3, 1)$ which correspond to the opposite (though equal) characteristics $(8, -4, 16)$, $(8, 4, 16)$, M. Arndt might have given the two forms $(1, 2, 0, -4)$ et $(1, -1, -3, -1)$ corresponding to the *same* characteristic $(8, 4, 16)$; but then it would not have appeared at a glance that the two classes

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were opposite to each other; and I presume that it is for this reason that he has selected the two representative forms $(1, -1, -3, -1)$ and $(1, 1, -3, 1)$. It must not, however, be imagined that the opposite cubic forms which correspond to opposite characteristics, which are ambiguous (and therefore equal to each other), are always, as in the last preceding example, unequal: for example Det. -144, there is only the form $(0, -2, -2, 3)$ given as corresponding to the ambiguous characteristic $(8, 4, 20)$; the opposite form $(0, 2, -2, -3)$ corresponding to the opposite but equal characteristic $(8, -4, 20)$ is equal to $(0, -2, -2, 3)$, and so does not give rise to a distinct opposite class. In the new tables, the sign $\pm$ is only employed in regard to opposite ambiguous characteristics; for instance, Det -4×28 there are given (not included in a single expression by means of the sign $\pm$) the two forms $(1, -1, -3, 1)$, $(1, 1, -3, 1)$ corresponding to the characteristic $2\,(2, \pm 1, 4)$.

I remark, that in a few instances M. Arndt, in passing from the second to the third column, has modified the expression for a cubic form in such manner that the characteristic has ceased to be a reduced form; for instance, Det. -216, he has given in the third column the two forms $(0, \mp 3, 2, \pm 1)$ belonging to the characteristic $(18, \mp 6, 14)$; it would have been better, it appears to me, to preserve the expression of the second column $(1, -2, -3, 0)$, and adopt the two representative forms $(1, \mp 2, -3, 0)$; I have accordingly made this change.

I divide M. Arndt's table into two tables; the first of them corresponding to the determinants $\equiv 0 \pmod 4$, the second to the determinants $\equiv 1 \pmod 4$. In the first table I take for the characteristic the form

$$\{b^2 - ac,\ \tfrac{1}{2}(bc - ad),\ bd - c^2\},$$

which belongs to the determinant $-\frac{1}{4}D$, and I arrange the cubic classes according to their *order;* viz. we have the properly primitive order (pp) when the terms $(a, 3b, 3c, d)$ have no common divisor; and the improperly primitive order (ip) when the terms $(a, 3b, 3c, d)$ have no common divisor other than 3, or what is the same thing when a and d being each of them divisible by 3, the terms (a, b, c, d) have no common divisor. But, moreover, the characteristic $\{b^2 - ac,\ \frac{1}{2}(bc - ad),\ bd - c^2\}$, may be of the properly primitive order pp; or of the improperly primitive order ip; or it may be of a derived order $\mu\,(A', B', C'), = \mu.pp$ or $\mu.ip$, according as (A', B', C') considered as a form belonging to

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the determinant $B^2 - A'C', = -\frac{1}{4\mu^2} D$ is of the properly primitive or the improperly primitive order. And in these different cases, the cubic class is said to be of the order *pp* on *pp*, *pp* on *ip*, *pp* on $\mu.pp$, *pp* on $\mu.ip$, *ip* on *pp*, &c., as the case may be.

For the determinants $\equiv 1 \pmod 4$, I retain the characteristic

$$\{2(b^2 - ac),\ bc - ad,\ 2(c^2 - bd)\},$$

and this being so, the division into orders is the same as in the former case; only as the characteristic, when primitive, is of necessity improperly primitive, the orders *pp* on *pp* and *ip* on *pp* no longer exist.

To every characteristic I annex in the tables the symbol of its composition; viz., 1 denotes the principal forms, c a form which by its duplication, d a form which by its triplication, &c., produces the principal form, σ denotes the most simple form of order *ip*, σc, σd, &c., the forms obtained by combining σ with the forms c, d, &c., of the order *pp*. Similarly to a characteristic $\mu(A', B', C')$ I annex the symbol of composition of the form (A', B', C'), (considered as belonging to the determinant $B^2 - A'C', = -\frac{D}{4\mu^2}$) multiplying this symbol by the number μ; for instance, $\mu.1$ denotes that (A', B', C') is the principal form, and similarly in other cases.

I have given a third table for the determinants

$$-4 \times 243,\ -4 \times 307,\ -4 \times 339,\ -4 \times 459,\ -4 \times 675,$$

where -243, -307, -339, -459, -675 are those of the thirteen irregular negative determinants in the first thousand for which the number of classes is divisible by 3. The number -4×675, $= -2700$, is beyond the limits of M. Arndt's Table, but the calculation (at least for the order *pp* on *pp*) presents no difficulty.

I remark, that according to M. Arndt (*Grunert*, t. XVII., p. 19), the number of cubic forms corresponding to a given characteristic (A, B, C) is equal to the number of proper transformations of $(A, -B, C)$, Det. D, into $(\frac{1}{2}A^2, B^2 - \frac{1}{2}AC, \frac{1}{2}C^2)$, Det. DB^2, so that when there is no such transformation, there exists no cubic form corresponding to the characteristic (A, B, C). This includes, I believe, the theorem in a letter of mine to M. Hermite, *Quarterly*

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Mathematical Journal, t. I., (1855) p. 85, viz., that for a *pp* form (A, B, C) of negative determinant, there is either no corresponding cubic form, or else a single corresponding cubic form, according as (A, B, C) does not, or does, produce by its triplication the principal form; but the particular theorem, in the cases to which it applies, is the more convenient one: it shows at once that for a regular negative determinant the number of cubic forms corresponding to a properly primitive characteristic (or, what is the same thing, number of cubic classes of the order (*pp* or *ip*) on *pp*) is 1 or 3, according as the number of quadratic classes is not, or is, divisible by 3.

The inspection of the tables gives rise to other remarks, but at present I abstain from pursuing the subject further; I will only notice that in some instances for example Det. -224, the classes which correspond to characteristics of the principal genus are partly of the order *pp* on *pp* and partly of the order *ip* on *pp*.

Table I. *of the binary cubic forms, the determinants of which are the negative numbers* $\equiv 0$ (*mod.* 4) *from* -4 *to* -400.

Det.		Order			
$4\times$	Classes.		on	Charact.	Compn.
1	0, −1, 0, 1	*pp*	*pp*	1, 0, 1	1
2	0, −1, 0, 2	*pp*	*pp*	1, 0, 2	1
3	0, −1, 0, 3	*ip*	*pp*	1, 0, 3	1
4	0, −1, 0, 4	*pp*	*pp*	1, 0, 4	1
	1, −1, −1, 1	*pp*	2*pp*	2 (1, 0, 1)	2.1
5	0, −1, 0, 5	*pp*	*pp*	1, 0, 5	1
6	0, −1, 0, 6	*ip*	*pp*	1, 0, 6	1
7	0, −1, 0, 7	*pp*	*pp*	1, 0, 7	1
	1, 0, −2, 2 1, 0, −2, −2	*pp*	2*ip*	2 (2, ± 1, 4)	2σ
8	0, −1, 0, 8	*pp*	*pp*	1, 0, 8	1
	0, −2, 0, 1	*pp*	2*pp*	2 (1, 0, 2)	2.1
9	0, −1, 0, 9	*ip*	*pp*	1, 0, 9	1
10	0, −1, 0, 10	*pp*	*pp*	1, 0, 10	1

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Det. $4\times$	Classes.	Order	on	Charact.	Compn.
11	0, −1, 0, 11	*pp*	*pp*	1, 0, 11	1
	0, −2, −1, 1			3, 1, 4	d
	0, −2, 1, 1			3, −1, 4	d^2
12	0, −1, 0, 12	*ip*	*pp*	1, 0, 12	1
13	0, −1, 0, 13	*pp*	*pp*	1, 0, 13	1
14	0, −1, 0, 14	*pp*	*pp*	1, 0, 14	1
15	0, −1, 0, 15	*ip*	*pp*	1, 0, 15	1
	1, −2, 0, 2	*pp*	*ip*	4, ±1, 4	σc
	1, 2, 0, −2				
16	0, −1, 0, 16	*pp*	*pp*	1, 0, 16	1
17	0, −1, 0, 17	*pp*	*pp*	1, 0, 17	1
18	0, −1, 0, 18	*ip*	*pp*	1, 0, 18	1
	1, 1, −2, −2	*pp*	3*pp*	3 (1, 0, 2)	3.1
	1, −1, −2, 2				
19	0, −1, 0, 19	*pp*	*pp*	1, 0, 19	1
	0, 2, 1, −2			4, 1, 5	d
	0, −2, 1, 2			4, −1, 5	d^2
20	0, −1, 0, 20	*pp*	*pp*	1, 0, 20	1
	0, −2, −2, 1	*pp*	2*pp*	2 (2, 1, 3)	2c
21	0, −1, 0, 1	*ip*	*pp*	1, 0, 21	1
22	0, −1, 0, 22	*pp*	*pp*	1, 0, 22	1
23	0, −1, 0, 23	*pp*	*pp*	1, 0, 23	1
	1, −1, −2, 4			3, −1, 8	d^2
	1, 1, −2, −4			3, 1, 8	d
24	0, −1, 0, 24	*ip*	*pp*	1, 0, 24	1
	0, −2, 0, 3	*ip*	2*pp*	2 (2, 0, 3)	2c
25	0, −1, 0, 25	*pp*	*pp*	1, 0, 25	1
	1, −2, −1, 2	*pp*	5*pp*	5 (1, 0, 1)	5.1
	1, 2, −1, −2				

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Det. $4\times$	Classes.	Order	on	Charact.	Compn.
26	0, −1, 0, 26			1, 0, 26	1
	1, 0, −3, 2	*pp*	*pp*	3, −1, 9	g^2
	1, 0, −3, −2			3, 1, 9	g^4
27	0, −1, 0, 27			1, 0, 27	1
	0, −2, 1, 3	*ip*	*pp*	4, −1, 7	d^2
	0, 2, 1, −3			4, 1, 7	d
	0, −3, 0, 1	*pp*	3*pp*	3 (1, 0, 3)	3.1
28	0, −1, 0, 28	*pp*	*pp*	1, 0, 28	1
	1, 1, −3, 1	*pp*	2*ip*	2 (2, ± 1, 4)	2σ
	1, −1, −3, −1				
29	0, −1, 0, 29			1, 0, 29	1
	2, 1, −2, −2	*pp*	*pp*	5, 1, 6	g^2
	2, −1, −2, 2			5, −1, 6	g^4
30	0, −1, 0, 30	*ip*	*pp*	1, 0, 30	1
31	0, −1, 0, 31			1, 0, 31	1
	2, 1, −2, −3	*pp*	*pp*	5, 2, 7	d
	2, −1, −2, 3			3, −2, 7	d^2
32	0, −1, 0, 32	*pp*	*pp*	1, 0, 32	1
33	0, −1, 0, 33	*ip*	*pp*	1, 0, 33	1
34	0, −1, 0, 34	*pp*	*pp*	1, 0, 34	1
35	0, −1, 0, 35			1, 0, 35	1
	0, 2, 1, −4	*pp*	*pp*	4, 1, 9	g^2
	0, −2, 1, 4			4, −1, 9	g^4
36	0, −1, 0, 36	*ip*	*pp*	1, 0, 36	1
	0, −2, −2, 3	*ip*	2*pp*	2 (2, 1, 5)	$2c$
37	0, −1, 0, 37	*pp*	*pp*	1, 0, 37	1
38	0, −1, 0, 38			1, 0, 38	1
	2, −2, −1, 3	*ip*	*pp*	6, 2, 7	g^2
	2, 2, −1, −3			6, −2, 7	g^4
39	0, −1, 0, 39	*ip*	*pp*	1, 0, 39	1

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Det. $4\times$	Classes.	Order		Charact.	Compn.
53	0, −1, 0, 53			1, 0, 53	1
	1, −3, 0, 2	*pp*	*pp*	6, −1, 9	g^3
	1, 3, 0, −2			6, 1, 9	g^4
54	0, −1, 0, 54	*ip*		1, 0, 54	1
	1, 2, −3, 0	*pp*	*pp*	7, −3, 9	g^4
	1, −2, −3, 0	*pp*		7, 3, 9	g^2
	0, −3, 0, 2	*pp*	3*pp*	3 (2, 0, 3)	3*c*
	0, 3, 0, −2				
55	0, −1, 0, 55	*pp*	*pp*	1, 0, 55	1
	1, −1, −3, 5	*pp*	*ip*	4, −1, 14	σe^3
	1, 1, −3, −5			4, 1, 14	σe
56	0, −1, 0, 56	*pp*	*pp*	1, 0, 56	1
	0, −2, 0, 7	*pp*	2*pp*	2 (2, 0, 7)	$2e^2$
57	0, −1, 0, 57	*ip*	*pp*	1, 0, 57	1
58	0, −1, 0, 58	*pp*	*pp*	1, 0, 58	1
59	0, −1, 0, 59			1, 0, 59	1
	0, −2, 1, 7	*pp*	*pp*	4, −1, 15	j^6
	0, 2, 1, −7			4, 1, 15	j^3
60	0, −1, 0, 60	*ip*		1, 0, 60	1
	1, 0, −4, 4	*pp*	2*ip*	2 (2, ±1, 8)	2σ
	1, 0, −4, −4				
61	0, −1, 0, 61			1, 0, 61	1
	1, −2, −1, 6	*pp*	*pp*	5, −2, 13	g^2
	1, 2, −1, −6			5, 2, 13	g^4
62	0, −1, 0, 62	*pp*	*pp*	1, 0, 62	1
63	0, −1, 0, 63	*ip*	*pp*	1, 0, 63	1
	1, 0, −4, 2	*pp*	*ip*	4, −1, 16	σe
	1, 0, −4, −2			4, 1, 16	σe^3
64	0, −1, 0, 64	*pp*	*pp*	1, 0, 64	1
	1, 0, −4, 0	*pp*	4*pp*	4 (1, 0, 4)	4.1
65	0, −1, 0, 65	*pp*	*pp*	1, 0, 65	1

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Det. $4\times$	Classes.	Order	on	Charact.	Compn.
81	0, −1, 0, 81	*pp*		1, 0, 81	1
	0, −3, 2, 2	*ip*	*pp*	9, −3, 10	g^2
	0, 3, 2, −2	*ip*		9, 3, 10	g^4
82	0, −1, 0, 82	*pp*	*pp*	1, 0, 82	1
83	0, −1, 0, 83			1, 0, 83	1
	0, −2, 1, 10	*pp*	*pp*	4, −1, 21	j^3
	0, 2, 1, −10			4, 1, 21	j^3
84	0, −1, 0, 84	*ip*	*pp*	1, 0, 84	1
	0, −2, −2, 9	*ip*	2*pp*	2 (2, 1, 11)	2σ
85	0, −1, 0, 85	*pp*	*pp*	1, 0, 85	1
86	0, −1, 0, 86	*pp*	*pp*	1, 0, 86	1
87	0, −1, 0, 87	*ip*		1, 0, 87	1
	1, −2, −3, 2	*pp*	*pp*	7, −2, 13	g^4
	1, 2, −3, −2	*pp*		7, 2, 13	g^3
88	0, −1, 0, 88	*pp*	*pp*	1, 0, 88	1
	0, −2, 0, 11	*pp*	2*pp*	2 (2, 0, 11)	2σ
89	0, −1, 0, 89			1, 0, 89	1
	1, −1, −4, 2	*pp*	*pp*	5, −1, 18	m^3
	1, 1, −4, −2			5, 1, 18	m^4
90	0, −1, 0, 90	*ip*	*pp*	1, 0, 90	1
91	0, −1, 0, 91			1, 0, 91	1
	0, −2, 1, 11	*pp*	*pp*	4, −1, 23	g^4
	0, 2, 1, −11			4, 1, 23	g^2
92	0, −1, 0, 92			1, 0, 92	1
	2, −3, 0, 4	*pp*	*pp*	9, −4, 12	g^4
	2, 3, 0, −4			9, 4, 12	g^2
93	0, −1, 0, 93	*ip*	*pp*	1, 0, 93	1
94	0, −1, 0, 94	*p*	*pp*	1, 0, 94	1

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Ten.		Cyc.
1 1	[illegible]	
66	1, — 1, 1, 66	2
67	1, — 1, 1, 67	
	1, — 1, 1, 4	11
	1, 1, 1, — 4	
68	1, — 1, 1, 34	
	1, — 1, — 1, 7	
69	1, — 1, 1, 23	
70	0, — 1, 1, 70	
71	0, — 1, 1, 71	
	1, — 1, 1, 3	
	1, 1, 1, — 3	
72	0, — 1, 1, 72	
	0, — 1, 1, 9	
	1, — 1, — 1, 4	
	1, 1, — 1, —	
	2, — 3, 0, .	
	2, 3, 0, —	
73	0, — 1,	
74	0, — 1	
75	0, —	
	0, .	
	0,	
76		

xd
xd^2

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		Order			
	as.		on	Charact.	Compn.
	1, 6	*ip*	*ip*	2, 1, 14	σ
	2, −1				
	−2, 1	*pp*	$3ip$	$3(2, \pm 1, 2)$	3σ
	1, −1, 2				
	1, −1, 7			2, 1, 16	σ
	1, −2, 1	*pp*	*ip*	4, −1, 8	σd
	0, −2, −1			4, 1, 8	σd^2
	, −1, −1, 8	*pp*	*ip*	2, 1, 18	σ
	0, −1, −1, 9	*ip*	*ip*	2, 1, 20	σ
	0, −1, −1, 10	*pp*	*ip*	2, 1, 22	σ
	0, −1, −1, 11	*pp*	*ip*	2, 1, 24	σ
	0, −1, −1, 12	*ip*	*ip*	2, 1, 26	σ
55	0, −1, −1, 13	*pp*	*ip*	2, 1, 28	σ
59	0, −1, −1, 14			2, 1, 30	σ
	1, −1, −2, 1	*pp*	*ip*	6, 1, 10	σj
	1, 1, −2, −1			6, −1, 10	σj^2
63	0, −1, −1, 15	*ip*	*ip*	2, 1, 32	σ
67	0, −1, −1, 16	*pp*	*ip*	2, 1, 34	σ
71	0, −1, −1, 17	*pp*	*ip*	2, 1, 36	σ
75	0, −1, −1, 18	*ip*	*ip*	2, 1, 38	σ
79	0, −1, −1, 19	*pp*	*ip*	2, 1, 40	σ
83	0, −1, −1, 20			2, 1, 42	σ
	1, −1, −2, 3	*pp*	*ip*	6, −1, 14	σj^2
	1, 1, −2, −3			6, 1, 14	σj
87	0, −1, −1, 21	*ip*		2, 1, 44	σ
	1, −2, 0, 3	*pp*	*ip*	8, −3, 12	σg^3
	1, 2, 0, −3	*pp*		8, 3, 12	σg^4
91	0, −1, −1, 22	*pp*	*ip*	2, 1, 46	σ
95	0, −1, −1, 23	*pp*	*ip*	2, 1, 48	σ

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Det. $4\times$	Classes.	Order	on	Charact.	Compn.
95	0, −1, 0, 95	*pp*	*pp*	1, 0, 95	1
	1, −2, −2, 6	*pp*	*ip*	6, −1, 16	σi^7
	1, 2, −2, −6	*pp*	*ip*	6, 1, 16	σi
96	0, −1, 0, 96	*ip*	*pp*	1, 0, 96	1
97	0, −1, 0, 97	*pp*	*pp*	1, 0, 97	1
98	0, −1, 0, 98	*pp*	*pp*	1, 0, 98	1
99	0, −1, 0, 99	*ip*	*pp*	1, 0, 99	1
	0, −2, 1, 12	*ip*	*pp*	4, −1, 25	g^3
	0, 2, 1, −12	*ip*	*pp*	4, 1, 25	g^4
100	0, −1, 0, 100	*pp*	*pp*	1, 0, 100	1
	0, −2, −2, 11	*pp*	2*pp*	2 (2, 1, 13)	2.1
	1, −1, −4, 4	*pp*	5*pp*	5 (1, 0, 4)	5.1
	1, 1, −4, −4	*pp*	5*pp*	5 (1, 0, 4)	5.1
	1, −3, −1, 3	*pp*	10*pp*	10 (1, 0, 1)	10.1
	1, 3, −1, −3	*pp*	10*pp*	10 (1, 0, 1)	10.1

Table II. *of the binary cubic forms the determinants of which are the positive numbers* $\equiv 1$ (*mod.* 4) *from* −3 *to* −99.

Det. $4\times$	Classes.	Order	on	Charact.	Compn.
3	0, 1, 1, 0	*ip*	*ip*	2, ±1, 2	σ
	1, 0, −1, 1	*pp*	*ip*	2, ±1, 2	σ
	1, 0, −1, −1	*pp*	*ip*	2, ±1, 2	σ
7	0, −1, −1, 1	*pp*	*ip*	2, 1, 4	σ
11	0, −1, −1, 2	*pp*	*ip*	2, 1, 6	σ
15	0, −1, −1, 3	*ip*	*ip*	2, 1, 8	σ
19	0, −1, −1, 4	*pp*	*ip*	2, 1, 10	σ
23	0, −1, −1, 5	*pp*	*ip*	2, 1, 12	σ
	1, −1, −1, 2	*pp*	*ip*	4, −1, 6	σd
	1, 1, −1, −2	*pp*	*ip*	4, 1, 6	σd^2

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Det. $4\times$	Classes.	Order on	on	Charact.	Compn.
27	$0, -1, -1, 6$	ip	ip	2, 1, 14	σ
	$1, 1, -2, -1$ $1, -1, -2, 1$ $2, -1, -1, 2$	pp	$3ip$	$3(2, \pm 1, 2)$	3σ
31	$0, -1, -1, 7$			2, 1, 16	σ
	$1, 0, -2, 1$	pp	ip	$4, -1, 8$	σd
	$1, 0, -2, -1$			4, 1, 8	σd^2
35	$0, -1, -1, 8$	pp	ip	2, 1, 18	σ
39	$0, -1, -1, 9$	ip	ip	2, 1, 20	σ
43	$0, -1, -1, 10$	pp	ip	2, 1, 22	σ
47	$0, -1, -1, 11$	pp	ip	2, 1, 24	σ
51	$0, -1, -1, 12$	ip	ip	2, 1, 26	σ
55	$0, -1, -1, 13$	pp	ip	2, 1, 28	σ
59	$0, -1, -1, 14$			2, 1, 30	σ
	$1, -1, -2, 1$	pp	ip	6, 1, 10	σj
	$1, 1, -2, -1$			$6, -1, 10$	σj^2
63	$0, -1, -1, 15$	ip	ip	2, 1, 32	σ
67	$0, -1, -1, 16$	pp	ip	2, 1, 34	σ
71	$0, -1, -1, 17$	pp	ip	2, 1, 36	σ
75	$0, -1, -1, 18$	ip	ip	2, 1, 38	σ
79	$0, -1, -1, 19$	pp	ip	2, 1, 40	σ
83	$0, -1, -1, 20$			2, 1, 42	σ
	$1, -1, -2, 3$	pp	ip	$6, -1, 14$	σj^2
	$1, 1, -2, -3$			6, 1, 14	σj
87	$0, -1, -1, 21$	ip		2, 1, 44	σ
	$1, -2, 0, 3$	pp	ip	$8, -3, 12$	σg^2
	$1, 2, 0, -3$	pp		8, 3, 12	σg^4
91	$0, -1, -1, 22$	pp	ip	2, 1, 46	σ
95	$0, -1, -1, 23$	pp	ip	2, 1, 48	σ

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Det. $4\times$	Classes.	Order	on	Charact.	Compn.
99	0, −1, −1, 24	*ip*	*ip*	2, 1, 50	σ
	1, 0, −3, 3	*pp*	$3ip$	$3(2, \pm 1, 6)$	3σ
	1, 0, −3, −3				

Table III. *of the binary cubic forms the determinants of which are the negative numbers* −972, −1228, 1336, −1836 *et* −2700. (−4 × 675 = −2700 is beyond Arndt's Tables).

Det. $4\times$	Classes.	Order	on	Charact.	Compn.
243	0, −1, 0, 243	*ip*		1, 0, 243	1
	1, −1, −6, 0	*pp*		7, 3, 36	d
	1, 1, −6, 0	*pp*		7, −3, 36	d^2
	0, 2, 1, −30	*ip*		4, 1, 61	d_1
	2, 3, −2, −5	*pp*	*pp*	13, −2, 19	dd_1
	0, 3, 2, −8	*pp*		9, 3, 28	d^2d_1
	0, −2, 1, 30	*ip*		4, −1, 61	d_1^2
	0, −3, 2, 8	*pp*		9, −3, 28	dd_1^2
	2, −3, −2, 5	*pp*		13, 2, 19	$d^2d_1^2$
307	0, −1, 0, 307			1, 0, 307	1
	1, 1, −6, −8			7, 1, 44	d
	1, −1, −6, 8			7, −1, 44	d^2
	0, 2, 1, −38			4, 1, 77	d_1
	1, −3, −2, 8	*pp*	*pp*	11, −1, 28	dd_1
	4, 1, −4, −3			17, 4, 19	d^2d_1
	0, −2, 1, 38			4, −1, 77	d_1^2
	4, −1, −4, 3			17, −4, 19	dd_1^2
	1, 3, −2, 8			11, 1, 28	$d^2d_1^2$
339	0, −1, 0, 339	*ip*		1, 0, 339	1
	1, 0, −7, −4	*pp*		7, 2, 49	d
	1, 0, −7, 4	*pp*		7, −2, 49	d^2
	0, 2, 1, −42	*ip*		4, 1, 85	d_1
	3, 0, −5, −4	*pp*	*pp*	15, 6, 25	dd_1
	2, −3, −2, 8	*pp*		13, −5, 28	d^2d_1
	0, −2, 1, 42	*ip*		4, −1, 85	d_1^2
	2, 3, −2, −8	*pp*		13, 5, 28	dd_1^2
	3, 0, −5, 4	*pp*		15, −6, 25	$d^2d_1^2$

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Det. $4\times$	Classes.	Order	on	Charact.	Compn.
459	0, −1, 0, 459	*ip*		1, 0, 459	1
	0, 3, 2, −16	*pp*		9, 3, 52	d
	0, −3, 2, 16	*pp*		9, −3, 52	d^2
	0, 2, 1, −57	*ip*		4, 1, 115	d_1
	1, 4, −3, −4	*pp*	*pp*	19, −4, 25	dd_1
	2, −1, 6, 0	*pp*		13, 3, 36	d^2d_1
	0, −2, 1, 57	*ip*		4, −1, 115	d_1^2
	2, 1, −6, 0	*pp*		13, −3, 36	dd_1^2
	1, −4, −3, 4	*pp*		19, 4, 25	$d^2d_1^2$
	0, −3, 0, 17	*pp*	3*pp*	3 (3, 0, 17)	$3g^2$
675	0, −1, 0, 675			1, 0, 675	1
	0, 3, 2, −24			9, 3, 76	d
	0, −3, 2, 24			9, −3, 76	d^2
	0, 2, 1, −168			4, 1, 169	d_1
	0, 5, −4, −3	*ip*	*pp*	25, −10, 31	dd_1
	3, −1, −6, 0			19, 3, 36	d^2d_1
	0, −1, 1, 168			4, −1, 169	d_1^2
	3, 1, −6, 0			19, −3, 36	dd_1^2
	0, −5, −4, 3			25, 10, 31	$d^2d_1^2$

N.B. For this last determinant -4×675, there may possibly be other cubic classes based on a non-primitive characteristic; I have not ascertained whether such forms do or do not exist.

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ON THE TRANSFORMATION OF TWO SIMULTANEOUS EQUATIONS.

By WILLIAM WALTON, M.A., Fellow of Trinity Hall.

THE object of this article is to prove that the two simultaneous equations

$$\frac{a(b-c)}{a-\alpha}+\frac{b(c-a)}{b-\beta}+\frac{c(a-b)}{c-\gamma}=0 \ldots\ldots\ldots (1),$$

$$\frac{\alpha(\beta-\gamma)}{a-\alpha}+\frac{\beta(\gamma-\alpha)}{b-\beta}+\frac{\gamma(\alpha-\beta)}{c-\gamma}=0\ldots\ldots\ldots(2),$$

which are satisfied by either of the two systems

$$\left.\begin{matrix} b-c=\beta-\gamma \\ c-a=\gamma-\alpha \\ a-b=\alpha-\beta \end{matrix}\right\}, \quad \left\{\frac{a}{\alpha}=\frac{b}{\beta}=\frac{c}{\gamma}\right\},$$

and *sub modo* by each of the three systems

$$(b=\beta,\ c=\gamma),$$

$$(c=\gamma,\ a=\alpha),$$

$$(a=\alpha,\ b=\beta);$$

can be satisfied by no other independent system.

Put $a-\alpha=u$, $b-\beta=v$, $c-\gamma=w$; then (1) becomes

$$\frac{a}{u}(b-c)+\frac{b}{v}(c-a)+\frac{c}{w}(a-b)=0 \ldots\ldots\ldots (3).$$

Again, (2) is equivalent to

$$\frac{a(\beta-\gamma)}{a-\alpha}+\frac{b(\gamma-\alpha)}{b-\beta}+\frac{c(\alpha-\beta)}{c-\gamma}=0,$$

and therefore, by (1), we have

$$\frac{a}{a-\alpha}\{(b-\beta)-(c-\gamma)\}+\frac{b}{b-\beta}\{(c-\gamma)-(a-\alpha)\}$$
$$+\frac{c}{c-\gamma}\{(a-\alpha)-(b-\beta)\}=0,$$

and consequently

$$\frac{a}{u}(v-w)+\frac{b}{v}(w-u)+\frac{c}{w}(u-v)=0\ldots\ldots\ldots(4).$$

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Multiply the equations (3), (4), and the identical equation

$$\frac{1}{u}.u(v-w)+\frac{1}{v}.v(w-u)+\frac{1}{w}.w(u-v)=0$$

by λ, μ, ν, respectively, add together the resulting equations, and equate to zero the coefficients of $\frac{1}{u}$, $\frac{1}{v}$, $\frac{1}{w}$, and we have

$$\begin{aligned}
&\lambda a(b-c)+\mu a(v-w)+\nu u(v-w)=0,\\
&\lambda b(c-a)+\mu b(w-u)+\nu v(w-u)=0,\\
&\lambda c(a-b)+\mu c(u-v)+\nu w(u-v)=0;
\end{aligned}$$

whence, by addition,

$$u(b-c)+v(c-a)+w(a-b)=0 \ldots\ldots\ldots (5).$$

The equation (3) may be written in the form

$$\begin{aligned}
&u\{c(a-b)v+b(c-a)w\}\\
&+v\{a(b-c)w+c(a-b)u\}\\
&+w\{b(c-a)u+a(b-c)v\}=0,
\end{aligned}$$

and therefore in the form

$$\begin{aligned}
&u\left(\frac{w-v}{a}+\frac{v}{b}-\frac{w}{c}\right)\\
&+v\left(\frac{u-w}{b}+\frac{w}{c}-\frac{u}{a}\right)\\
&+w\left(\frac{v-u}{c}+\frac{u}{a}-\frac{v}{b}\right)=0 \ldots\ldots\ldots (6).
\end{aligned}$$

Multiplying the identical equation

$$u(v-w)+v(w-u)+w(u-v)=0,$$

and the equations (5), (6), respectively by 1, λ, μ, adding, and equating to zero the coefficients of u, v, w, we have

$$\left.\begin{aligned}
&v-w+\lambda(b-c)+\mu\left(\frac{w-v}{a}+\frac{v}{b}-\frac{w}{c}\right)=0\\
&w-u+\lambda(c-a)+\mu\left(\frac{u-w}{b}+\frac{w}{c}-\frac{u}{a}\right)=0\\
&u-v+\lambda(a-b)+\mu\left(\frac{v-u}{c}+\frac{u}{a}-\frac{v}{b}\right)=0
\end{aligned}\right\} \ldots (7).$$

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"that if two concentric and similarly situated quadrics intersect in a common line of curvature, they are confocal."

Let, in fact, the equations to two confocal quadrics be

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \ldots\ldots\ldots\ldots\ldots (14),$$

$$\frac{x^2}{a-\eta} + \frac{y^2}{b-\eta} + \frac{z^2}{c-\eta} = 1 \ldots\ldots\ldots\ldots\ldots (15);$$

and suppose that the quadric

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{z^2}{\gamma} = 1 \ldots\ldots\ldots\ldots\ldots (16),$$

intersects (14) in the line of curvature of (14) in which (14) and (15) intersect.

From (14) and (16) we have

$$\left(\frac{1}{a} + \frac{\lambda}{\alpha}\right)x^2 + \left(\frac{1}{b} + \frac{\lambda}{\beta}\right)y^2 + \left(\frac{1}{c} + \frac{\lambda}{\gamma}\right)z^2 = \lambda + 1.$$

Then, under our supposition, we must have

$$\left.\begin{aligned} \frac{1+\lambda}{a-\eta} &= \frac{1}{a} + \frac{\lambda}{\alpha} \\ \frac{1+\lambda}{b-\eta} &= \frac{1}{b} + \frac{\lambda}{\beta} \\ \frac{1+\lambda}{c-\eta} &= \frac{1}{c} + \frac{\lambda}{\gamma} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (17).$$

Eliminating λ and η between the equations (17) we shall easily ascertain that

$$\frac{a(b-c)}{a-\alpha} + \frac{b(c-a)}{b-\beta} + \frac{c(a-b)}{c-\gamma} = 0,$$

as the condition that the intersection of (14) and (16) shall be a line of curvature of (14).

Similarly the condition that the intersection of (14) and (16) shall be a line of curvature of (16) is

$$\frac{\alpha(\beta-\gamma)}{a-\alpha} + \frac{\beta(\gamma-\alpha)}{b-\beta} + \frac{\gamma(\alpha-\beta)}{c-\gamma} = 0.$$

Of the systems (12) and (13), deduced above from these two equations, the former shews that (14) and (16) are confocal; the latter shews that (14) and (16) are either coincident surfaces or that they do not intersect.

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Observe, however, as regards the system of the three solutions, that if for instance the quadric surfaces are

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1,$$

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{\gamma}=1;$$

these surfaces are not confocal but they intersect (touch) in the ellipse $z=0$, $\frac{x^2}{a}+\frac{y^2}{b}=1$, which is a principal section, and consequently a curve of curvature, of each of them; and similarly for the other two of the three solutions.

Note by Prof. Cayley.

Writing in Mr. Walton's equations (1) and (2)

$$\frac{a}{\delta}, \frac{b}{\delta}, \frac{c}{\delta}, \frac{\alpha}{\delta}, \frac{\beta}{\delta}, \frac{\gamma}{\delta}$$

instead of a, b, c, α, β, γ respectively; and putting for shortness

$$A=b\gamma-c\beta,\quad F=a\delta-d\alpha,$$

$$B=c\alpha-a\gamma,\quad G=b\delta-d\beta,$$

$$C=a\beta-b\alpha,\quad H=c\delta-d\gamma,$$

the equations become

$$\frac{a(b-c)}{F}+\frac{b(c-a)}{G}+\frac{c(a-b)}{H}=0,$$

$$\frac{\alpha(\beta-\gamma)}{F}+\frac{\beta(\gamma-\alpha)}{G}+\frac{\gamma(\alpha-\beta)}{H}=0.$$

Multiplying by FGH and effecting some obvious transformations, the equations become

$$aAF+bBG+cCH=0 \quad\ldots\ldots\ldots\ldots\ldots(18),$$

$$\alpha AF+\beta BG+\gamma CH=0;$$

whence also

$$AF^2+BG^2+CH^2=0 \quad\ldots\ldots\ldots\ldots\ldots(19).$$

Now regarding $(\alpha, \beta, \gamma, \delta)$ as the coordinates of a point in space, the equations (18) and (19) represent each of them a cone having for vertex the point $\alpha:\beta:\gamma:\delta=a:b:c:d$,

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viz. (18) is a quadric cone, (19) a cubic cone; they intersect therefore in six lines; and it may be shown that these are

the line	$\alpha : \beta : \gamma = a : b : c$	(twice)	2
„	$\beta : \gamma : \delta = b : c : d$		1
„	$\gamma : \alpha : \delta = c : a : d$		1
„	$\alpha : \beta : \delta = a : b : d$		1
„	$\beta-\gamma : \gamma-\alpha : \alpha-\beta : \delta = b-c : c-a : a-b : d$		$\frac{1}{6}$

agreeing with Mr. Walton's result.

ON RICCATI'S EQUATION.

By J. W. L. Glaisher, B.A.

In the sixteenth *Cahier* of the *Journal de l'école Polytechnique*, Poisson has shown that if

$$y = \int_0^\infty e^{-x^n - \frac{ba^n}{x^n}}\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

then y satisfies the differential equation

$$\frac{d^2y}{da^2} = n^2 b a^{n-2} y \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which is well known as the form of Riccati's equation

$$\frac{dt}{da} + t^2 = n^2 b a^{n-2},$$

when transformed by assuming

$$t = \frac{1}{y}\frac{dy}{da}.$$

From this result, by quoting the conditions of integrability of Riccati's equation, Poisson infers that the definite integral (1) can be evaluated in a finite form whenever $n = \frac{2}{1 \pm 2i}$,

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i being zero or an integer; but, with the exception of remarking that (1) affords a *particular* integral of (2) in the form of a definite integral he makes no use of his investigation to obtain the solution of Riccati's equation. The condition of integrability, however, of this equation,* and the general integral when it admits of expression in a finite form can be obtained as follows:

Taking Riccati's equation in the form in which Prof. Cayley has written it in the *Philosophical Magazine*, Vol. II., p. 348, for 1868, we have

$$\frac{dy}{dx} + y^2 = x^{2q-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

which, on assuming $y = \frac{1}{w}\frac{dw}{dx}$ becomes

$$\frac{d^2w}{dx^2} - x^{2q-2}w = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

so that if $w = C_1P + C_2Q$ is the general integral of (4), C_1 and C_2 being the arbitrary constants, then

$$y = \frac{P' + CQ'}{P + CQ}$$

is the general integral of (3), where $P' = \frac{dP}{dx}$, $Q' = \frac{dQ}{dx}$, and C is the constant of integration. From Poisson's investigation we know that

$$w = \int_0^\infty e^{-z^{2q} - \frac{x^{2q}}{4q^2z^{2q}}} dz \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

is a particular integral of (4); put $z^{2q} = \alpha v^2$, α being a constant, and (5) takes the form

$$w = \frac{\alpha^{\frac{1}{2q}}}{q} \int_0^\infty v^{\frac{1}{q}-1} e^{-\alpha v^2 - \frac{x^{2q}}{4q^2\alpha v^2}} dv.$$

Take

$$\frac{x^{2q}}{4q^2} = \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

* The form of n for which (2) admits of a particular integral expressible in finite terms is deduced from (1) by De Morgan, *Diff. and Int. Calc.*, p. 704; but neither the particular nor general integrals of Riccati's Equation, to which the method leads, are deduced.

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and we obtain

$$w = \int_0^\infty v^{\frac{1}{q}-1} e^{-av^q - \frac{1}{v^2}}\, dv,$$

leaving out the constant multiplier. Now

$$\int_0^\infty e^{-av^2 - \frac{1}{v^2}}\, dv = \frac{\sqrt{\pi}}{2\sqrt{a}}\, e^{-2\sqrt{a}},$$

so that $$\int_0^\infty v^{2i} e^{-av^2 - \frac{1}{v^2}}\, dv = (-)^i \frac{\sqrt{\pi}}{2} \left(\frac{d}{da}\right)^i \frac{e^{-2\sqrt{a}}}{\sqrt{a}},$$

i being positive; if then we take $\frac{1}{q} - 1 = 2i$, that is $q = \frac{1}{2i+1}$, we have a particular integral of (4) expressed in a finite form, viz.

$$w = x \left(\frac{d}{da}\right)^i \frac{e^{-2\sqrt{a}}}{\sqrt{a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

where a is given by (6), so that

$$\frac{d}{da} = \frac{1}{2q}\, x^{-2q+1} \frac{d}{dx},$$

whence $$w = x \left(x^{-2q+1} \frac{d}{dx}\right)^i \frac{e^{-\frac{x^q}{q}}}{x^q} \quad \ldots\ldots\ldots\ldots\ldots (8).$$

Now if $y = xf(x^q)$ is a solution of (4), so also is $y = xf(-x^q)$,* (*Phil. Mag.* loc. cit.), so that when $q = \frac{1}{2i+1}$, the general integral of (4) is

$$w = x \left(x^{-2q+1} \frac{d}{dx}\right)^i \frac{C_1 e^{-\frac{1}{q}x^q} + C_2 e^{\frac{1}{q}x^q}}{x^q} \quad \ldots\ldots\ldots\ldots (9),$$

C_1 and C_2 being arbitrary constants.

Similarly, we can obtain the solution if $q = -\frac{1}{2i+1}$, for putting $x^q = v$ in (5), and taking

$$\frac{x^{2q}}{4q^2} = \beta \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

* This can be shown by substituting in the differential equation (4). Similarly if $w = f(x^q)$ is a solution, so also is $w = f(-x^q)$. The change of sign of x^q in (8) is best effected by changing the sign of $\sqrt{a}$ in (7).

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we have as a particular integral

$$w = \int_0^\infty v^{\frac{1}{q}-1} e^{-v^q - \frac{\beta}{v^q}} dv,$$

which, if $\frac{1}{q} - 1 = -2i - 2$, that is if $q = -\frac{1}{2i+1}$,

$$= (-)^{i+1} \left(\frac{d}{d\beta}\right)^{i+1} \int_0^\infty e^{-v^q - \frac{\beta}{v^q}} dv$$

$$= (-)^{i+1} \frac{\sqrt{\pi}}{2} \left(\frac{d}{d\beta}\right)^{i+1} e^{-2\sqrt{\beta}},$$

whence substituting for β and $\frac{d}{d\beta}$ from (10), we obtain the particular integral

$$w = \left(x^{-2q+1} \frac{d}{dx}\right)^{i+1} e^{-\frac{1}{q}x^q},$$

and the general integral

$$w = \left(x^{-2q+1} \frac{d}{dx}\right)^{i+1} \left(C_1 e^{-\frac{1}{q}x^q} + C_2 e^{\frac{1}{q}x^q}\right) \ldots\ldots (11),$$

when $$q = -\frac{1}{2i+1}.$$

The expressions (9) and (11) give the solution of Riccati's equation in the only two integrable cases; in (9) i may be any positive integer or zero, in (11) i may be any positive integer, zero, or -1.

It should be noticed that since

$$x^{-2q+1} \frac{d}{dx} e^{-\frac{x^q}{q}} = -x^{-q} e^{-\frac{x^q}{q}},$$

the formula (9) may also be written

$$w = x \left(x^{-2q+1} \frac{d}{dx}\right)^{i+1} \left(C_1 e^{-\frac{x^q}{q}} + C_2 e^{\frac{x^q}{q}}\right),$$

which is a form more nearly resembling (11).

If in (9) we put $i = 1$ we obtain

$$w = C_1 \left(-\tfrac{1}{3} - x^{\frac{1}{3}}\right) e^{-2x^{\frac{1}{3}}} + C_2 \left(-\tfrac{1}{3} + x^{\frac{1}{3}}\right) e^{2x^{\frac{1}{3}}},$$

as the solution of

$$\frac{d^2w}{dx^2} - x^{-\frac{4}{3}} w = 0,$$

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and, by putting $i=2$,

$$w = C_1 (\tfrac{8}{35} + \tfrac{8}{5}x^{\frac{1}{2}} + x^{\frac{3}{2}})\, e^{-\frac{4}{5}x^{\frac{5}{2}}} + C_2 (\tfrac{8}{35} - \tfrac{8}{5}x^{\frac{1}{2}} + x^{\frac{3}{2}})\, e^{\frac{4}{5}x^{\frac{5}{2}}},$$

as the solution of

$$\frac{d^2w}{dx^2} - x^{-\frac{3}{2}}w = 0,$$

and if in (11) we put $i=1$, we obtain

$$w = C_1 (x^{\frac{3}{2}} - \tfrac{1}{3}x)\, e^{2x^{-\frac{1}{2}}} + C_2 (x^{\frac{3}{2}} + \tfrac{1}{3}x)\, e^{-2x^{-\frac{1}{2}}},$$

which agree with Prof. Cayley's solutions in the *Philosophical Magazine.*

By equating the values of w in (9) and (11) to those given in the *Philosophical Magazine*, and properly determining the constants C_1 and C_2, we obtain the theorems.

If $q(2i+1)=1$, then

$$x\left(x^{-2q+1}\frac{d}{dx}\right)^i (x^{-q}e^{-\frac{1}{q}x^q}) = (-)^i \frac{q(q-1)\,2q(2q-1)\ldots iq(iq-1)}{(q-1)(3q-1)\ldots\{(2i-1)q-1\}}$$

$$\left\{1 + \frac{q-1}{q(q-1)}x^q + \frac{(q-1)(3q-1)}{q(q-1)\,2q(2q-1)}x^{2q} + \ldots\right\} e^{-\frac{1}{q}x^q} \ldots (12),$$

$$x\left(x^{-2q+1}\frac{d}{dx}\right)^i (x^{-q}e^{\frac{1}{q}x^q}) = (-)^i \frac{q(q-1)\,2q(2q-1)\ldots iq(iq-1)}{(q-1)(3q-1)\ldots\{(2i-1)q-1\}}$$

$$\left\{1 - \frac{q-1}{q(q-1)}x^q + \frac{(q-1)(3q-1)}{q(q-1)\,2q(2q-1)}x^{2q} - \ldots\right\} e^{\frac{1}{q}x^q} \ldots\ldots (13),$$

and if $q(2i+1) = -1$,

$$\left(x^{-2q+1}\frac{d}{dx}\right)^{i+1} e^{-\frac{1}{q}x^q} = (-)^{i+1}\, x\, \frac{(q+1)\,q\,(2q+1)\,2q\ldots(iq+1)\,iq}{(q+1)(3q+1)\ldots\{(2i-1)q+1\}}$$

$$\left\{1 + \frac{q+1}{(q+1)q}x^q + \frac{(q+1)(3q+1)}{(q+1)\,q\,(2q+1)\,2q}x^{2q} + \ldots\right\} e^{-\frac{1}{q}x^q},$$

$$\left(x^{-2q+1}\frac{d}{dx}\right)^{i+1} e^{\frac{1}{q}x^q} = (-)^i\, x\, \frac{(q+1)\,q\,(2q+1)\,2q\ldots(iq+1)\,iq}{(q+1)(3q+1)\ldots\{(2i-1)q+1\}}$$

$$\left\{1 - \frac{q+1}{(q+1)q}x^q + \frac{(q+1)(3q+1)}{(q+1)\,q\,(2q+1)\,2q}x^{2q} - \ldots\right\} e^{\frac{1}{q}x^q}.$$

The first two of these theorems may be verified in the following manner:

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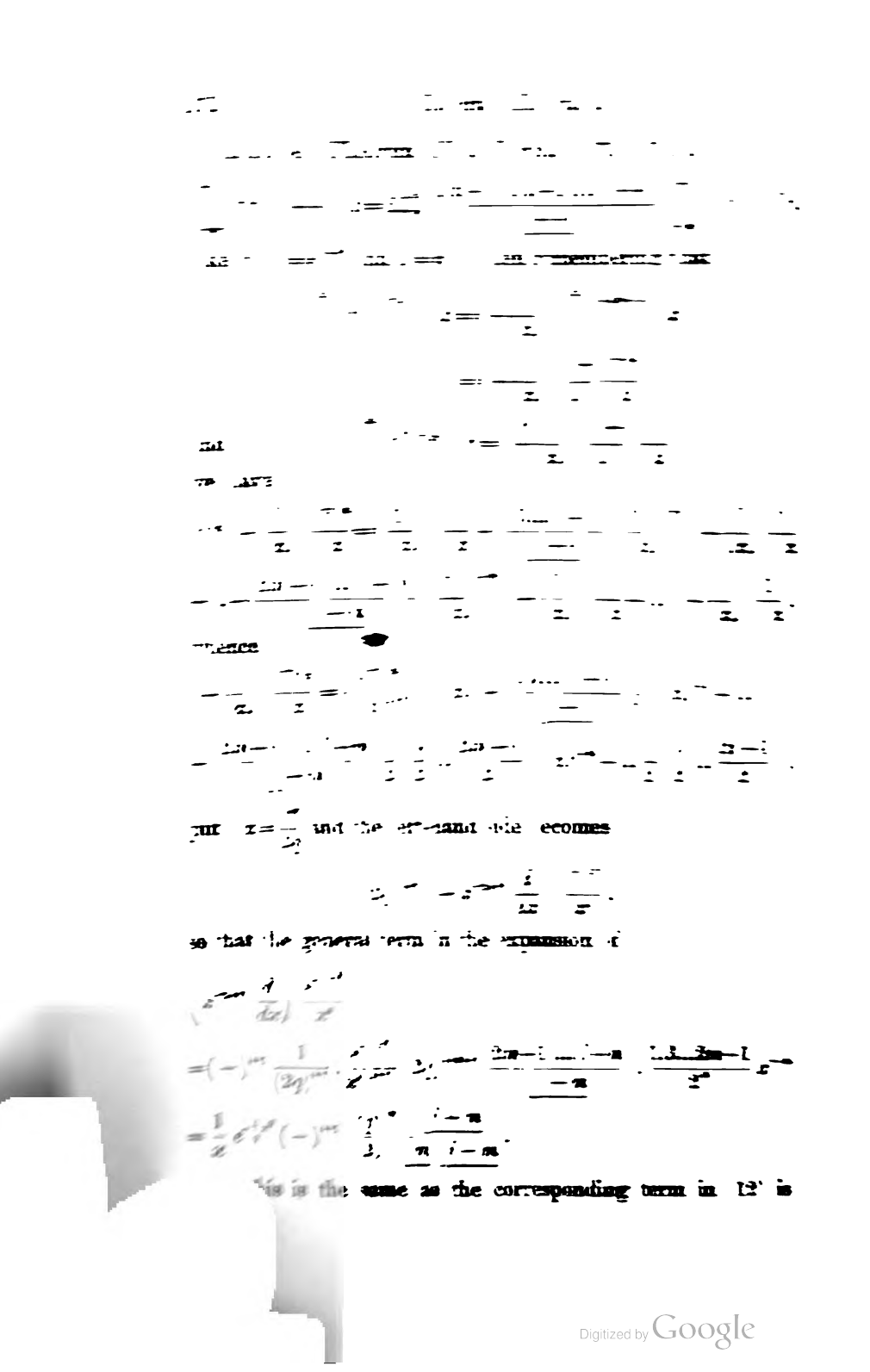

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easily seen, for the coefficient of the latter, on writing $q=\frac{1}{2i+1}$ so that $nq-1=(n-2i-1)q$, becomes

$$(-)^i q^m \frac{(-i-1)\,i\ldots(-i-m)(i-m+1)}{-2\ldots-2m}$$

$$=(-)^{i+1}\left(\frac{q}{2}\right)^m \frac{\lfloor i+m}{\lfloor m \lfloor i-m}.$$

By changing the sign of x^{-q}, (13) is also verified.

In (12) and (13) we can, as before, replace

$$x\left(x^{-2q+1}\frac{d}{dx}\right)^i (x^{-q}e^{-\frac{1}{q}x^q}) \text{ and } x\left(x^{-2q+1}\frac{d}{dx}\right)^i (x^{-q}e^{\frac{1}{q}x^q})$$

by $-x\left(x^{-2q+1}\frac{d}{dx}\right)^{i+1}e^{-\frac{1}{q}x^q}$ and $-\left(x^{-2q+1}\frac{d}{dx}\right)^{i+1}e^{\frac{1}{q}x^q}$ respectively.

In connection with the integral

$$\int_0^\infty e^{-\alpha x^2-\frac{\beta}{x^2}}\,dx=\frac{\sqrt{\pi}}{2}\,\frac{e^{-2\sqrt{(\alpha\beta)}}}{\sqrt{\alpha}},$$

two results easily obtained from it may be noticed,

$$\int_0^\infty x^{2i}e^{-\alpha x^2-\frac{\beta}{x^2}}\,dx=\frac{\sqrt{\pi}}{2}\left(-\frac{d}{d\alpha}\right)^i\frac{e^{-2\sqrt{(\alpha\beta)}}}{\sqrt{\alpha}},$$

and also

$$=\frac{\sqrt{\pi}}{2}\left\{\int^\infty d\beta\right\}^i\frac{e^{-2\sqrt{(\alpha\beta)}}}{\sqrt{\alpha}},$$

whence

$$\left\{\int^\infty d\beta\right\}^i e^{-2\sqrt{(\alpha\beta)}}=\sqrt{\alpha}\left(-\frac{d}{d\alpha}\right)^i\frac{e^{-2\sqrt{(\alpha\beta)}}}{\sqrt{\alpha}}$$

$$=\sqrt{\alpha}\left(-\frac{d}{d\alpha}\right)^{i+1}e^{-2\sqrt{(\alpha\beta)}}.$$

Similarly, by equating the two values of

$$\int_0^\infty \frac{1}{x^{2i}}\,e^{-\alpha x^2-\frac{\beta}{x^2}}\,dx,$$

we have

$$\left\{\int^\infty d\alpha\right\}^i\frac{e^{-2\sqrt{(\alpha\beta)}}}{\sqrt{\alpha}}=\frac{1}{\sqrt{\alpha}}\left(-\frac{d}{d\beta}\right)e^{-2\sqrt{(\alpha\beta)}},$$

or

$$\sqrt{\alpha}\left\{\int^\infty d\alpha\right\}^{i-1}e^{-2\sqrt{(\alpha\beta)}}=(-)^{i-1}\left(\frac{d}{d\beta}\right)^i e^{-2\sqrt{(\alpha\beta)}}.$$

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$$+\frac{(2r-1)(2r-2)(2r-3)(2r-4)}{1.2.3.4}\,l^{2r-5}m^4\frac{d^{2r}P}{du^{2r-5}dv^5}$$
$$+\ \ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\frac{2r-1}{1}\,lm^{2r-2}\frac{d^{2r}P}{dudv^{2r-1}}$$
$$+m^{2r-1}\frac{d^{2r}P}{dv^{2r}}:$$

hence, by the formulæ given in page 179 of this Volume of the *Journal*,

$$\left(l\frac{d}{du}+m\frac{d}{dv}\right)^{2r-1}\frac{dP}{dv}$$
$$=(-1)^1\,l^{2r-1}\frac{d^{2r}Q}{du^{2r}}$$
$$+(-1)^1\frac{2r-1}{1}\,l^{2r-2}m\frac{d^{2r}P}{du^{2r}}$$
$$+(-1)^2\frac{(2r-1)(2r-2)}{1.2}\,l^{2r-3}m^2\frac{d^{2r}Q}{du^{2r}}$$
$$+(-1)^2\frac{(2r-1)(2r-2)(2r-3)}{1.2.3}\,l^{2r-4}m^3\frac{d^{2r}P}{du^{2r}}$$
$$+(-1)^3\frac{(2r-1)(2r-2)(2r-3)(2r-4)}{1.2.3.4}\,l^{2r-5}m^4\frac{d^{2r}Q}{du^{2r}}$$
$$+\ \ldots\ldots\ldots\ldots\ldots\ldots$$
$$+(-1)^r\frac{2r-1}{1}\,lm^{2r-2}\frac{d^{2r}Q}{du^{2r}}$$
$$+(-1)^rm^{2r-1}\frac{d^{2r}P}{du^{2r}}.$$

Again

$$\left(l\frac{d}{du}+m\frac{d}{dv}\right)^{2r-1}\frac{dP}{du}=l^{2r-1}\frac{d^{2r}P}{du^{2r}}$$
$$+\frac{2r-1}{1}\,l^{2r-2}m\frac{d^{2r}P}{du^{2r-1}dv}$$
$$+\frac{(2r-1)(2r-2)}{1.2}\,l^{2r-3}m^2\frac{d^{2r}P}{du^{2r-2}dv^2}$$
$$+\frac{(2r-1)(2r-2)(2r-3)}{1.2.3}\,l^{2r-4}m^3\frac{d^{2r}P}{du^{2r-3}dv^3}$$
$$+\ \ldots\ldots\ldots\ldots\ldots\ldots$$

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$$+\frac{(2r-1)(2r-2)}{1.2}\,l^2m^{2r-3}\,\frac{d^{2r}P}{du^2dv^{2r-3}}$$

$$+\frac{2r-1}{1}\,lm^{2r-2}\,\frac{d^{2r}P}{du^2dv^{2r-2}}$$

$$+m^{2r-1}\,\frac{d^{2r}P}{dudv^{2r-1}}$$

$$=l^{2r-1}\,\frac{d^{2r}P}{du^{2r}}+(-1)^1\,\frac{2r-1}{1}\,l^{2r-2}m\,\frac{d^{2r}Q}{du^{2r}}$$

$$+(-1)^2\,\frac{(2r-1)(2r-2)}{1.2}\,l^{2r-3}m^2\,\frac{d^{2r}P}{du^{2r}}$$

$$+(-1)^3\,\frac{(2r-1)(2r-2)(2r-3)}{1.2.3}\,l^{2r-4}m^3\,\frac{d^{2r}Q}{du^{2r}}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+(-1)^{r-1}\,\frac{(2r-1)(2r-2)}{1.2}\,l^2m^{2r-3}\,\frac{d^{2r}Q}{du^{2r}}$$

$$+(-)^{r-1}\,\frac{2r-1}{1}\,lm^{2r-2}\,\frac{d^{2r}P}{du^{2r}}$$

$$+(-1)^r m^{2r-1}\,\frac{d^{2r}Q}{du^{2r}}.$$

Hence

$$\left(l\frac{d}{du}+m\frac{d}{dv}\right)^{2r-1}\left(l\frac{d}{dv}-m\frac{d}{du}\right)P$$

$$=\frac{d^{2r}Q}{du^{2r}}\left\{-l^{2r}+\frac{2r(2r-1)}{1.2}\,l^{2r-2}m^2-\frac{2r(2r-1)(2r-2)(2r-3)}{1.2.3.4}\,l^{2r-4}m^4\right.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+(-1)^r\,\frac{2r(2r-1)}{1.2}\,l^2m^{2r-2}$$

$$\left.-(-1)^r m^{2r}\right\}$$

$$+\frac{d^{2r}P}{du^{2r}}\left\{-\frac{2r}{1}\,l^{2r-1}m+\frac{2r(2r-1)(2r-2)}{1.2.3}\,l^{2r-3}m^3\right.$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left.+(-1)^r\,\frac{2r}{1}\,lm^{2r-1}\right\}$$

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$$= -\tfrac{1}{2}\frac{d^{2r}Q}{du^{2r}}\left[\{l+m\sqrt{(-1)}\}^{2r}+\{l-m\sqrt{(-1)}\}^{2r}\right]$$

$$-\frac{1}{2\sqrt{(-1)}}\frac{d^{2r}P}{du^{2r}}\left[\{l+m\sqrt{(-1)}\}^{2r}-\{l-m\sqrt{(-1)}\}^{2r}\right].$$

But, θ being the inclination of the tangent of any branch of the multiple point to the axis of u, $l=\cos\theta$, $m=\sin\theta$; hence

$$\left(l\frac{d}{du}+m\frac{d}{dv}\right)^{2r-1}\left(l\frac{d}{dv}-m\frac{d}{du}\right)P$$

$$=-\frac{d^{2r}Q}{du^{2r}}\cos 2r\theta-\frac{d^{2r}P}{du^{2r}}\sin 2r\theta.$$

Again

$$\left(l\frac{d}{du}+m\frac{d}{dv}\right)^{2r+1}P$$

$$=l^{2r+1}\frac{d^{2r+1}P}{du^{2r+1}}+\frac{2r+1}{1}l^{2r}m\frac{d^{2r+1}P}{du^{2r}dv}$$

$$+\frac{(2r+1).2r}{1.2}l^{2r-1}m^{2}\frac{d^{2r+1}P}{du^{2r-1}dv^{2}}$$

$$+\frac{(2r+1).2r.(2r-1)}{1.2.3}l^{2r-2}m^{3}\frac{d^{2r+1}P}{du^{2r-2}dv^{3}}$$

$$+\frac{(2r+1).2r.(2r-1).(2r-2)}{1.2.3.4}l^{2r-3}m^{4}\frac{d^{2r+1}P}{du^{2r-3}dv^{4}}$$

$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{(2r+1).2r.(2r-1)}{1.2.3}l^{3}m^{2r-2}\frac{d^{2r+1}P}{du^{3}dv^{2r-2}}$$

$$+\frac{(2r+1).2r}{1.2}l^{2}m^{2r-1}\frac{d^{2r+1}P}{du^{2}dv^{2r-1}}$$

$$+\frac{2r+1}{1}lm^{2r}\frac{d^{2r+1}P}{dudv^{2r}}$$

$$+m^{2r+1}\frac{d^{2r+1}P}{dv^{2r+1}}$$

$$=l^{2r+1}\frac{d^{2r+1}P}{du^{2r+1}}+(-1)^{1}\frac{2r+1}{1}l^{2r}m\frac{d^{2r+1}Q}{du^{2r+1}}$$

$$+(-1)^{1}\frac{(2r+1).2r}{1.2}l^{2r-1}m^{2}\frac{d^{2r+1}P}{du^{2r+1}}$$

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Hence, by the formula for the curvature of the branches at the multiple point, we have

$$\frac{1}{\rho} = \frac{1}{r(2r+1)} \cdot \frac{\frac{d^{2r+1}P}{du^{2r+1}} \cos(2r+1)\theta - \frac{d^{2r+1}Q}{du^{2r+1}} \sin(2r+1)\theta}{\frac{d^{2r}Q}{du^{2r}} \cos 2r\theta + \frac{d^{2r}P}{du^{2r}} \sin 2r\theta}.$$

But, (see page 92 of this Volume of the *Journal*),

$$\frac{\sin 2r\theta}{\cos 2r\theta} = \frac{\frac{d^{2r}P}{du^{2r}}}{\frac{d^{2r}Q}{du^{2r}}};$$

hence

$$\frac{1}{\rho} = \frac{1}{r(2r+1)} \cdot \frac{\frac{d^{2r+1}P}{du^{2r+1}}\left(\frac{d^{2r}Q}{du^{2r}}\cos\theta - \frac{d^{2r}P}{du^{2r}}\sin\theta\right) - \frac{d^{2r+1}Q}{du^{2r+1}}\left(\frac{d^{2r}P}{du^{2r}}\cos\theta + \frac{d^{2r}Q}{du^{2r}}\sin\theta\right)}{\left(\frac{d^{2r}P}{du^{2r}}\right)^2 + \left(\frac{d^{2r}Q}{du^{2r}}\right)^2}$$

$$= \frac{1}{r(2r+1)} \cdot \frac{\cos\theta\left(\frac{d^{2r+1}P}{du^{2r+1}}\frac{d^{2r}Q}{du^{2r}} - \frac{d^{2r+1}Q}{du^{2r+1}}\frac{d^{2r}P}{du^{2r}}\right) - \sin\theta\left(\frac{d^{2r}P}{du^{2r}}\frac{d^{2r+1}P}{du^{2r+1}} + \frac{d^{2r}Q}{du^{2r}}\frac{d^{2r+1}Q}{du^{2r+1}}\right)}{\left(\frac{d^{2r}P}{du^{2r}}\right)^2 + \left(\frac{d^{2r}Q}{du^{2r}}\right)^2}$$

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If we next suppose the multiplicity to be odd and proceed in the same way, putting $\nu = 2r+1$, we shall arrive at the result

$$\frac{1}{\rho} = \frac{1}{(2r+1)(r+1)} \cdot \frac{\cos\theta \left(\frac{d^{\nu+1}P}{du^{\nu+1}} \frac{d^{\nu+1}Q}{du^{\nu+1}} - \frac{d^{\nu+1}Q}{du^{\nu+1}} \frac{d^{\nu+1}P}{du^{\nu+1}} \right) - \sin\theta \left(\frac{d^{\nu+1}P}{du^{\nu+1}} \frac{d^{\nu+1}P}{du^{\nu+1}} + \frac{d^{\nu+1}Q}{du^{\nu+1}} \frac{d^{\nu+1}Q}{du^{\nu+1}} \right)}{\left(\frac{d^{\nu+1}P}{du^{\nu+1}} \right)^2 + \left(\frac{d^{\nu+1}Q}{du^{\nu+1}} \right)^2}.$$

Thus, at a point of ν^{th} multiplicity, whether even or odd, of the rhizic curve $P=0$,

$$\frac{1}{\rho} = \frac{2}{\nu(\nu+1)} \cdot \frac{\cos\theta \left(\frac{d^{\nu+1}P}{du^{\nu+1}} \frac{d^{\nu}Q}{du^{\nu}} - \frac{d^{\nu+1}Q}{du^{\nu+1}} \frac{d^{\nu}P}{du^{\nu}} \right) - \sin\theta \left(\frac{d^{\nu}P}{du^{\nu}} \frac{d^{\nu+1}P}{du^{\nu+1}} + \frac{d^{\nu}Q}{du^{\nu}} \frac{d^{\nu+1}Q}{du^{\nu+1}} \right)}{\left(\frac{d^{\nu}P}{du^{\nu}} \right)^2 + \left(\frac{d^{\nu}Q}{du^{\nu}} \right)^2}.$$

By precisely the same course of investigation we may prove that, at a point of ν^{th} multiplicity, whether even or odd, of the rhizic curve $Q=0$, $\frac{1}{\rho}$ is equal to the very same expression.

From the expression for $\frac{1}{\rho}$ it is evident that, σ and ϵ being constants for all the branches at the multiple point,

$$\sigma = \rho\cos(\theta+\epsilon),$$

and therefore, putting $\phi - \frac{\pi}{2}$ for θ,

$$\sigma = \rho\sin(\phi+\epsilon),$$

the equation to a straight line in which all the centres of curvature lie. This result establishes one of the two theorems, the demonstration of which is the object of this article.

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Imagine now that the two conjugate rhizic curves intersect at a multiple point, the multiplicity of each being, as is known, the same. Then, the equation

$$c = \rho \sin(\phi + \varepsilon)$$

being common to both curves, the centres of curvature at the common multiple point of both curves all lie in one and the same straight line. The values of ϕ for the curve $P=0$ differ by a constant quantity from the corresponding values of ϕ for the curve $Q=0$, and the centres of curvature of the two curves lie alternately in the common straight line.

By way of illustration, let

$$f(x) = x^4 + bx^3 - (6a^2 + 3ab)\,x^2 + (8a^3 + 3a^2b)\,x - 3a^4 - a^3b.$$

Then the equations to the two rhizic curves are

$$\begin{aligned} P = {} & u^4 - 6u^2v^2 + v^4 + bu^3 - 3buv^2 \\ & - (6a^2 + 3ab)(u^2 - v^2) \\ & + (8a^3 + 3a^2b)\,u - 3a^4 - a^3b, \\ Q = {} & v\{4u^3 + 3bu^2 - (12a^2 + 6ab)\,u + 8a^3 + 3a^2b\} \\ & - (4u + b)\,v^3 = 0. \end{aligned}$$

It may easily be ascertained that the two curves $P=0$, $Q=0$, have a common point $(u=a,\ v=0)$, a triple point of each.

At this point, in the curve $P=0$, we may find that, with respective reference to the three branches,

$$\left\{\begin{matrix}\theta = \dfrac{\pi}{6} \\ \rho = -3(4a+b)\end{matrix}\right\},\quad \left\{\begin{matrix}\theta = \dfrac{\pi}{2} \\ \rho = -\frac{3}{2}(4a+b)\end{matrix}\right\},\quad \left\{\begin{matrix}\theta = 2\pi - \dfrac{\pi}{6} \\ \rho = \frac{3}{2}(4a+b)\end{matrix}\right\}.$$

In relation to the curve $Q=0$,

$$\left\{\begin{matrix}\theta = \pm 0 \\ \rho = \mp \infty\end{matrix}\right\},\quad \left\{\begin{matrix}\theta = \dfrac{\pi}{3} \\ \rho = -\sqrt{3}\,(4a+b)\end{matrix}\right\},\quad \left\{\begin{matrix}\theta = 2\pi - \dfrac{\pi}{3} \\ \rho = \sqrt{3}\,(4a+b)\end{matrix}\right\}.$$

In the figure (34) the tangents at the triple point of $P=0$ are denoted by straight lines, those at the triple point of $Q=0$ by dotted lines, and the centres of curvature, each at a distance $\frac{3}{2}(4a+b)$ from the line $\theta = \dfrac{\pi}{2}$, by asterisks, the one infinitely distant not being indicated. The centres of curvature of each curve lie in the tangents to the branches of the other.

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(4) $XY=0$, $ZY'=0$, therefore $XZ=0$.

(5) $XY=0$, ZY' not $=0$.

(6) XY not $=0$, ZY' not $=0$.

And of these, there are (as shown by the third column) only two which give rise to a conclusion (or relation between the extreme terms). As regards the negative cases, this is at once seen to be so; thus $XY=0$, $ZY=0$ (no X's are Y's, no Z's are Y's) leads to no conclusion in regard to X, Z. As regards the positive cases, it is also at once seen that the conclusions do follow; but we may obtain the conclusions by symbolical reasoning, thus

(2) $Y=YX+YX'$, $=YX'$;

therefore $ZY=ZYX'$, not $=0$; therefore ZX' not $=0$.

(4) $XZ=XZY+XZY'$, where on the right-hand side each term (the first as containing XY, the second as containing ZX') is $=0$; that is, $XZ=0$; where the logical signification of each step is obvious.

ON THE INVERSION OF A QUADRIC SURFACE.

By Prof. CAYLEY.

THE inversion intended to be considered is that by reciprocal radius vectors, viz. if x, y, z are rectangular coordinates, and $r^2=x^2+y^2+z^2$, then x, y, z are to be changed into $\frac{x}{r^2}$, $\frac{y}{r^2}$, $\frac{z}{r^2}$. But it is convenient to introduce for homogeneity a fourth coordinate w, $=1$; and the change then is x, y, z into $\frac{xw^2}{r^2}$, $\frac{yw^2}{r^2}$, $\frac{zw^2}{r^2}$.

Starting from the quadric surface

$$(a, b, c, d, f, g, h, l, m, n)(x, y, z, w)^2=0,$$

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or what is the same thing

$$\begin{aligned}&(a, b, c, f, g, h\text{)}(x, y, z)^2\\ &+2w\,(lx+my+nz)\\ &+dw^2 \qquad\qquad =0,\end{aligned}$$

the equation of the inverse surface is

$$\begin{aligned}&w^2\,(a, b, c, f, g, h\text{)}(x, y, z)^2\\ &+2w\,(lx+my+nz)\,r^2\\ &+dr^4 \qquad\qquad =0,\end{aligned}$$

where $r^2=x^2+y^2+z^2$. The inverse surface is thus a quartic having the nodal conic $w=0$, $x^2+y^2+z^2=0$ (circle at infinity); and having the node $x=0$, $y=0$, $z=0$ (the centre of inversion); or say it is a nodal bicircular quartic surface, or nodal anallagmatic.

For x, y, z write $x-\frac{1}{2}\frac{l}{d}w$, $y-\frac{1}{2}\frac{m}{d}w$, $z-\frac{1}{2}\frac{n}{d}w$, and put for shortness

$$\begin{aligned}&lx+my+nz=u, \quad l^2+m^2+n^2=\alpha,\\ &al+hm+gn=\mathrm{a},\\ &hl+bm+fn=\mathrm{b}, \quad (a, b, c, f, g, h\text{)}(l, m, n)^2=A,\\ &gl+fm+cn=\mathrm{c},\end{aligned}$$

then r^2 becomes $r^2-\frac{uw}{d}+\frac{1}{4}\frac{\alpha}{d^2}w^2$,

$lx+my+nz$ „ $u-\frac{1}{2}\frac{\alpha}{d}w$,

$(a,\ldots)(x, y, z)^2$ „ $(a,\ldots)(x, y, z)^2-(\mathrm{a}x+\mathrm{b}y+\mathrm{c}z)\frac{w}{d}+\frac{1}{4}A\frac{w^2}{d^2}$.

Hence the equation is

$$\begin{aligned}&d\left\{r^4-2r^2\frac{uw}{d}+w^2\left(\frac{1}{2}\frac{\alpha}{d^2}r^2+\frac{u^2}{d^2}\right)-\frac{1}{2}\frac{\alpha uw^2}{d^3}+\frac{1}{16}\frac{\alpha^2}{d^4}w^4\right\}\\ &+2\left(wr^2-\frac{uw^2}{d}+\frac{1}{4}\frac{\alpha}{d^2}w^3\right)\left(u-\frac{1}{2}\frac{\alpha}{d}w\right)\\ &+w^2\left\{(a,\ldots)(x, y, z)^2-(\mathrm{a}x+\mathrm{b}y+\mathrm{c}z)\frac{w}{d}+\frac{1}{4}A\frac{w^2}{d^2}\right\}=0;\end{aligned}$$

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viz., arranging and reducing, this is

$$dr^4$$

$$+w^2\left\{-\frac{1}{2}\frac{\alpha}{d}r^2-\frac{u^2}{d}+(a,\ldots\chi x,y,z)^2\right\}$$

$$+w^3\left\{\frac{\alpha u}{d^2}-\frac{1}{d}(ax+by+cz)\right\}$$

$$+w^4\left\{-\frac{3}{16}\frac{\alpha^2}{d^2}+\tfrac{1}{4}A\frac{1}{d^2}\right\}\qquad=0;$$

and we may without loss of generality assume

$$-\frac{mn}{d}+f=0,\quad\text{that is}\quad df-mn=0,$$

$$-\frac{nl}{d}+g=0,\qquad\text{,,}\qquad dg-nl=0,$$

$$-\frac{lm}{d}+h=0,\qquad\text{,,}\qquad dh-lm=0.$$

The equation then is

$$r^4$$

$$+w^2\left\{-\frac{1}{2}\frac{\alpha}{d^2}(x^2+y^2+z^2)\right.$$

$$\left.+\left(\frac{a}{d}-\frac{l^2}{d^2}\right)x^2+\left(\frac{b}{d}-\frac{m^2}{d^2}\right)y^2+\left(\frac{c}{d}-\frac{n^2}{d^2}\right)z^2\right\}$$

$$+w^3\left\{\frac{\alpha u}{d^2}-\frac{1}{d^2}(ax+by+cz)\right\}$$

$$+w^4\left\{-\frac{3}{16}\frac{\alpha^2}{d^4}+\tfrac{1}{4}A\frac{1}{d^2}\right\}=0.$$

Write

$$ad-l^2=a'd,$$

$$bd-m^2=b'd,$$

$$cd-n^2=c'd.$$

We have

$$\mathfrak{a}=al+hm+gn=gn=al+\frac{lm^2}{d}+\frac{ln^2}{d}=\frac{l}{d}(ad-l^2+\alpha),$$

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that is $$a = a' - \frac{\alpha}{d},$$

and similarly $$b = mb' - \frac{m\alpha}{d},$$

$$c = nc' - \frac{n\alpha}{d}.$$

Whence also $$A = la' + m^2b' + n^2c' + \frac{\alpha^2}{d},$$

and the equation is

$$r^4$$

$$+ w^2 \left(-\frac{1}{2}\frac{\alpha}{d^2} + \frac{a'}{d} \right) x^2 + \left(-\frac{1}{2}\frac{\alpha}{d^2} + \frac{b'}{d} \right) y^2 + \left(-\frac{1}{2}\frac{\alpha}{d^2} + \frac{c'}{d} \right) z^2$$

$$+ w^2 \left(-\frac{la'}{d^2} x - \frac{mb'}{d^2} y - \frac{nc'}{d^2} z \right)$$

$$+ w^4 \left\{ \frac{1}{4d^2} (la' + m^2b' + n^2c') + \frac{1}{16}\frac{\alpha^2}{d^4} \right\} = 0.$$

This is Kummer's form, say

$$r^4 = 4w^2 \{\alpha_1 x^2 + \beta_1 y^2 + \gamma_1 z^2 + \delta_1 w^2 + 2w (a_1 x + b_1 y + c_1 z)\},$$

where

$$-4\alpha_1 = -\frac{1}{2}\frac{\alpha}{d^2} + \frac{a'}{d}, \qquad -8a_1 = -\frac{la'}{d^2},$$

$$-4\beta_1 = -\frac{1}{2}\frac{\alpha}{d^2} + \frac{b'}{d}, \qquad -8b_1 = -\frac{mb'}{d^2},$$

$$-4\gamma_1 = -\frac{1}{2}\frac{\alpha}{d^2} + \frac{c'}{d}, \qquad -8c_1 = -\frac{nc'}{d^2},$$

$$-4\delta_1 = \frac{1}{4d^2}(la' + m^2b' + n^2c') + \frac{1}{16}\frac{\alpha^2}{d^4}$$

Hence Kummer's equation

$$\delta_1 + \lambda^2 = \frac{a_1^2}{\lambda + \alpha_1} + \frac{b_1^2}{\lambda + \beta_1} + \frac{c_1^2}{\lambda + \gamma_1},$$

or say

$$64\delta_1 + 64\lambda^2 = \frac{256a_1^2}{4\lambda + 4\alpha_1} + \frac{256b_1^2}{4\lambda + 4\beta_1} + \frac{256c_1^2}{4\lambda + 4\gamma_1},$$

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becomes

$$64\lambda^2 - \frac{4}{d^2}(l^2a' + m^2b' + n^2c') - \frac{\alpha^2}{d^4}$$

$$= \frac{4l^2a'^2}{d^4\left(\frac{1}{2}\frac{\alpha}{d^2} - \frac{a'}{d} + 4\lambda\right)} + \frac{4m^2b'^2}{d^4\left(\frac{1}{2}\frac{\alpha}{d^2} - \frac{b'}{d} + 4\lambda\right)} + \frac{4n^2c'^2}{d^4\left(\frac{1}{2}\frac{\alpha}{d^2} - \frac{c'}{d} + 4\lambda\right)},$$

which is satisfied by $4\lambda = -\frac{1}{2}\frac{\alpha}{d^2}$. Write therefore

$$4\lambda + \frac{1}{2}\frac{\alpha}{d^2} = -\frac{\theta}{d},$$

that is

$$8\lambda = -\frac{2\theta}{d} - \frac{\alpha}{d^2}$$

$$64\lambda^2 = \frac{4\theta^2}{d^2} + \frac{4\theta\alpha}{d^3} + \frac{\alpha^2}{d^4}.$$

and the equation is

$$\frac{4\theta^2}{d^2} + \frac{4\theta\alpha}{d^3} - \frac{4}{d^2}(l^2a' + m^2b' + n^2c')$$

$$= \frac{4l^2a'^2}{d^4\left(-\frac{\theta}{d} - \frac{a'}{d}\right)} + \frac{4m^2b'^2}{d^4\left(-\frac{\theta}{d} - \frac{b'}{d}\right)} + \frac{4n^2c'^2}{d^4\left(-\frac{\theta}{d} - \frac{c'}{d}\right)},$$

viz. this is

$$l^2a' + m^2b' + n^2c' - \theta\alpha - \theta^2 d$$

$$= \frac{l^2a'^2}{\theta + a'} + \frac{m^2b'^2}{\theta + b'} + \frac{n^2c'^2}{\theta + c'},$$

which is of course satisfied by $\theta = 0$. Moreover the derived equation

$$-\alpha - 2\theta d = -\frac{l^2a'^2}{(\theta + a')^2} - \frac{m^2b'^2}{(\theta + b')^2} - \frac{n^2c'^2}{(\theta + c')^2}$$

is also satisfied by $\theta = 0$, so that this is a double root. The equation in fact is

$$\{\theta^2 d + \theta\alpha - (l^2a' + m^2b' + n^2c')\}(\theta + a')(\theta + b')(\theta + c')$$
$$+ \{l^2a'^2(\theta + b')(\theta + c') + m^2b'^2(\theta + c')(\theta + a') + n^2c'^2(\theta + a')(\theta + b')\} = 0,$$

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or, expanding and dividing by θ^3, this is

$$\begin{aligned} & d(\theta+a')(\theta+b')(\theta+c') \\ & +\alpha\{\theta^2+\theta(a'+b'+c')+b'c'+c'a'+a'b'\} \\ & -(l^2a'+m^2b'+n^2c')(\theta+a'+b'+c') \\ & +l^2a'^2+m^2b'^2+n^2c'^2=0, \end{aligned}$$

which gives the remaining three roots.

If $a'=b'=c'$ the equation is

$$(\theta+a'+\alpha)(\theta+a')^3=0.$$

I recall that we have

$$a,\ b,\ c,\ d,\ f=\frac{mn}{d},\ g=\frac{nl}{d},\ h=\frac{lm}{d},\ l,\ m,\ n,$$

$$a'=a-\frac{l^2}{d},\ b'=b-\frac{m^2}{d},\ c'=c-\frac{n^2}{d},\ \alpha=l^2+m^2+n^2,$$

so that the quadric surface is

$$d(a'x^2+b'y^2+c'z^2)+(lx+my+dz+dw)^2=0,$$

and that $\alpha_1, \beta_1, \gamma_1, \delta_1, a_1, b_1, c_1$ denoting as before, the equation of the inverse surface (referred to a different origin) is

$$r^4=4w^2\{\alpha_1x^2+\beta_1y^2+\gamma_1z^2+\delta_1w^2+2w(a_1x+b_1y+c_1z)\}.$$

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ON CERTAIN DEFINITE INTEGRALS.

By WILLIAM WALTON, M.A., Fellow of Trinity Hall.

THE object of this article is to give new and more simple demonstrations of properties of certain definite integrals which form the principal subject of the *Suite du Mémoire sur les Intégrales définies*, published by Poisson in the tenth volume of the *Journal Polytechnique*.

1. To evaluate the integral

$$y = \int_0^\pi \frac{(\sin x)^{2n} dx}{(1 - 2a\cos x + a^2)^n}, \text{ when } a < 1.$$

Poisson, in the first instance, puts

$$y = \int_0^\pi \frac{(\sin x)^p dx}{(1 - 2a\cos x + a^2)^n};$$

then, differentiating y with regard to a, and integrating by parts, he finally arrives at the differential equation

$$(1 - a^2)\frac{d^2y}{da^2} + \left\{\frac{p+1}{a} - a(4n + 1 - p)\right\}\frac{dy}{da} - 2n(2n - p)y = 0.$$

He then selects, as a particular case where integration can be effected, the value $2n$ for p: his equation then becomes

$$\frac{d^2y}{da^2} + \frac{2n+1}{a} \cdot \frac{dy}{da} = 0,$$

the integral of which is

$$y = C + C' \cdot a^{-2n}.$$

He then, in virtue of the general nature of the expansion of the expression for y under the integral sign, shews that $C' = 0$, and then, by equating a to zero, proves that $C = \int_0^\pi (\sin x)^{2n} dx$. Thus he shews that

$$\int_0^\pi \frac{(\sin x)^{2n} dx}{(1 - 2a\cos x + a^2)^n} = \int_0^\pi (\sin x)^{2n} dx.$$

Of course, if a be greater than unity, we at once infer from this result that

$$\int_0^\pi \frac{(\sin x)^{2n} dx}{(1 - 2a\cos x + a^2)^n} = \frac{1}{a^{2n}} \int_0^\pi (\sin x)^{2n} dx.$$

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The following method of establishing these results is more simple and points out more clearly the intrinsic origin of the simplicity of the case where $p = 2n$.

Put $$y = \int_0^\pi \frac{(\sin x)^{2n} dx}{\{(a - \cos x)^2 + \sin^2 x\}^n},$$

a being less than unity. Let $a - \cos x = r \cos\phi$, $\sin x = r \sin\phi$; then

$$(a - \cos x) \sin\phi = \sin x \cos\phi,$$

$$\sin(\phi + x) = a \sin\phi = a \sin\{(\phi + x) - x\},$$

whence $$\tan(\phi + x) = \frac{-a \sin x}{1 - a \cos x};$$

a result which shews that $\tan(\phi + x)$ is negative while x varies from 0 to π, and zero at both these limits. Hence, while x varies from 0 to π, ϕ varies from π to 0.

Thus we see that, ψ representing $\phi + x$,

$$y = \int_0^\pi (\sin\phi)^{2n} dx$$

$$= \int_{x=0}^{x=\pi} (\sin\phi)^{2n} d\psi - \int_{x=0}^{x=\pi} (\sin\phi)^{2n} d\phi$$

$$= \frac{1}{a^{2n}} \int_\pi^\pi (\sin\psi)^{2n} d\psi + \int_0^\pi (\sin\phi)^{2n} d\phi,$$

and therefore, the former term of the right-hand member of the equation being zero,

$$y = \int_0^\pi (\sin x)^{2n} dx.$$

2. To evaluate the integral

$$y = \int_0^\pi dx . \log(1 - 2a \cos x + a^2), \quad a < 1.$$

Poisson evaluates this integral by first differentiating the equation

$$\int_0^\pi \frac{(\sin x)^{2n} dx}{(1 - 2a \cos x + a^2)^n} = \int_0^\pi (\sin x)^{2n} dx,$$

with regard to n and afterwards putting $n = 0$. For the sake of verification he effects the evaluation also by the aid of the series

$$\frac{1 - a^2}{1 - 2a \cos x + a^2} = 1 + a \cos x + a^2 \cos 2x + a^3 \cos 3x + \ldots .$$

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The following method is easy and independent, and does not require the aid of series.

By the relations connecting x, ϕ, r, above given, we see that

$$y = 2\int_0^{\pi} dx \log r$$

$$= 2\int_{\phi}^{\pi} dx\,(\log \sin x - \log \sin \phi)$$

$$= 2\int_0^{\pi} \log \sin x . dx - 2\int_{x=0}^{x=\pi} \log \sin \phi\, d\psi$$

$$+ 2\int_{x=0}^{x=\pi} \log \sin \phi . d\phi.$$

But $$\int_{x=0}^{x=\pi} \log \sin \phi\, d\psi = \int_{\pi}^{\pi} \log\left(\frac{\sin \psi}{a}\right) . d\psi = 0,$$

and

$$2\int_{x=0}^{x=\pi} \log \sin \phi . d\phi = 2\int_{\pi}^{0} \log \sin \phi . d\phi = -2\int_0^{\pi} \log \sin x . dx;$$

and therefore $y = 0$.

If $a > 1$, of course it follows, from this result, that $y = 2\pi \log a$.

3. To give a demonstration of the following relation between two definite integrals, viz.

$$\int_0^{\pi} \frac{(\sin x)^p dx}{(1 - 2a\cos x + a^2)^n} = (1 - a^2)^{1+p-2n} \int_0^{\pi} \frac{(\sin x)^p dx}{(1 - 2a\cos x + a^2)^{1+p-n}}.$$

Poisson's demonstration of this "*rapport remarquable*" is grounded upon his primary differential equation between y and a, and is certainly abstruse.

The following investigation is comparatively easy.

By the aid of the relations between x, ϕ, r, we see that the relation is equivalent to

$$\int_0^{\pi} \frac{(\sin \phi)^p dx}{r^{2n-p}} = (1 - a^2)^{1+p-2n} \int_0^{\pi} \frac{(\sin \phi)^p dx}{r^{2+p-2n}}.$$

Now $$\int_0^{\pi} \frac{(\sin \phi)^p dx}{r^{2n-p}} = \int_{x=0}^{x=\pi} \frac{(\sin \phi)^p d\psi}{r^{2n-p}} - \int_{x=0}^{x=\pi} \frac{(\sin \phi)^p d\phi}{r^{2n-p}}$$

$$= \int_{\pi}^{\pi} \frac{(\sin \phi)^p d\psi}{r^{2n-p}} + \int_0^{\pi} \frac{(\sin \phi)^p d\phi}{r^{2n-p}}.$$

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But $\sin\phi = \frac{\sin\psi}{a}$, and r may be expressed in terms of $\sin\phi$ and therefore of $\sin\psi$: hence the former term of the right-hand member of the equation is zero: thus

$$\int_0^\pi \frac{(\sin\phi)^p\,dx}{r^{2n-p}} = \int_0^\pi \frac{(\sin\phi)^p\,d\phi}{r^{2n-p}}.$$

Similarly $$\int_0^\pi \frac{(\sin\phi)^p\,dx}{r^{2+p-2n}} = \int_0^\pi \frac{(\sin\phi)^p\,d\phi}{r^{2+p-2n}}.$$

Thus Poisson's relation is equivalent to

$$\int_0^\pi \frac{(\sin\phi)^p\,d\phi}{r^{2n-p}} = (1-a^2)^{1+p-2n}\int_0^\pi \frac{(\sin\phi)^p\,d\phi}{r^{2+p-2n}}.$$

From the relations between x, ϕ, r, we see that

$$r = a\cos\phi + (1-a^2\sin^2\phi)^{\frac{1}{2}},$$

and therefore $$\frac{1-a^2}{r} = -a\cos\phi + (1-a^2\sin^2\phi)^{\frac{1}{2}}.$$

Hence $$\int_0^\pi \frac{(\sin\phi)^p\,d\phi}{r^{2n-p}}$$

$$= \frac{1}{(1-a^2)^{2n-p}}\int_0^\pi (\sin\phi)^p\,d\phi\,\{-a\cos\phi + (1-a^2\sin^2\phi)^{\frac{1}{2}}\}^{2n-p}$$

$$= \frac{-1}{(1-a^2)^{2n-p}}\int_\pi^0 (\sin\omega)^p\,d\omega\,\{\;a\cos\omega + (1-a^2\sin^2\omega)^{\frac{1}{2}}\}^{2n-p},$$

where $\omega = \pi - \phi$,

$$= \frac{1}{(1-a^2)^{2n-p}}\int_0^\pi \frac{(\sin\phi)^p\,d\phi}{r^{p-2n}}.$$

But $$r - \frac{1-a^2}{r} = 2a\cos\phi:$$

hence $$\int_0^\pi \frac{(\sin\phi)^p\,d\phi}{r^{2n-p}} = (1-a^2)^{p-2n}\int_0^\pi \frac{r\,(\sin\phi)^p\,d\phi}{r^{1+p-2n}}$$

$$= (1-a^2)^{1+p-2n}\int_0^\pi \frac{(\sin\phi)^p\,d\phi}{r^{2+p-2n}}$$

$$+ \frac{2a}{(1-a^2)^{2n-p}}\int_0^\pi \frac{(\sin\phi)^p\cos\phi\,d\phi}{r^{1+p-2n}}$$

$$= (1-a^2)^{1+p-2n}\int_0^\pi \frac{(\sin\phi)^p\,d\phi}{r^{2+p-2n}}$$

$$+ \frac{2}{a^p(1-a^2)^{2n-p}}\int_\pi^\pi \frac{(\sin\psi)^p\cos\psi\,d\psi}{r^{1+p-2n}}.$$

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But the last term is evidently zero: hence

$$\int_0^\pi \frac{(\sin\phi)^p\,d\phi}{\gamma^{2n-p}} = (1-a^2)^{1+p-2n}\int_0^\pi \frac{(\sin\phi)^p\,d\phi}{\gamma^{2+p-2n}}.$$

If a be >1, then this relation must of course be replaced by

$$\int_0^\pi \frac{(\sin\phi)^p\,d\phi}{\gamma^{2n-p}} = (a^2-1)^{1+p-2n}\int_0^\pi \frac{(\sin\phi)^p\,d\phi}{\gamma^{2+p-2n}}.$$

4. To evaluate

$$\int_0^\pi \frac{x\sin x\,dx}{1+\cos^2 x}.$$

The evaluation of this integral is effected by Poisson in the same memoir by the aid of expansion. It may however be effected more simply in the following manner:

$$\int_0^\pi \frac{x\sin x\,dx}{1+\cos^2 x} = -\int_{x=0}^{x=\pi} \frac{x\,d.\cos x}{1+\cos^2 x}$$

$$= -x\tan^{-1}(\cos x) + \int \tan^{-1}(\cos x)\,dx, \text{ between limits,}$$

$$= \frac{\pi^2}{4} + \int_0^{\frac{1}{2}\pi} \tan^{-1}(\cos x)\,dx + \int_{\frac{1}{2}\pi}^{\pi} \tan^{-1}(\cos x)\,dx.$$

But $\int_{\frac{1}{2}\pi}^{\pi} \tan^{-1}(\cos x)\,dx = \int_{\frac{1}{2}\pi}^{0} \tan^{-1}(\cos x')\,dx'$, where $x' = \pi - x$,

$$= -\int_0^{\frac{1}{2}\pi} \tan^{-1}(\cos x)\,dx.$$

Hence

$$\int_0^\pi \frac{x\sin x\,dx}{1+\cos^2 x} = \frac{\pi^2}{4}.$$

June 19, 1871.

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ON THE THEORY OF THE CURVE AND TORSE.

By Professor CAYLEY.

THE fundamental relations in the theory of the Curve and Torse were first established in my "Memoire sur les Courbes à double courbure et sur les Surfaces developpables," *Liouv.* t. x. (1845), see also *Camb. and Dub. Math. Jour.*, t. IV. (1850), viz. I showed that the systems (m, r, β, h, n, y) and (r, n, m, x, α, g), (the notation is subsequently explained), each of them satisfied the Plückerian relations. An additional set of equations giving the values of $(\gamma, t, k, q, \gamma', t', k', q')$ was furnished by Dr. Salmon's "Theory of Reciprocal Surfaces," *Trans. R. I. Acad.*, t. XXIII. (1857), see also the *Solid Geometry*. The theory as thus established is complete in itself, but it does not take account of certain singularities υ, ω, H, G; the singularity υ was first considered in my paper "On a special sextic Developable," *Quart. Math. Jour.* t. VII. (1865), (there called θ), and I afterwards endeavoured to take account of the remaining singularities ω, H, G. I was in correspondence on the subject with Prof. Cremona, and the discovery of the complete forms of several of the formulæ is due to him.

There has recently appeared a very valuable memoir by M. Zeuthen, "Sur les singularités ordinaires d'une courbe gauche et d'une surface developpable," *Annali di Matem.* t. III. (1869); he excludes, however, from consideration the singularity ω, and does not throughout attend to H, G.

I propose in the present memoir to reproduce and develope the whole theory.

Explanations and Notation.

1. We have a singly infinite series of points, lines, and planes; viz. each line passes through two consecutive points and lies in two consecutive planes; each plane passes through three consecutive points and contains two consecutive lines; each point is the intersection of three consecutive planes and

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of two consecutive lines. The points describe and the lines envelope a *curve;* the lines describe and the planes envelope a *torse;* the entire system of points, lines, and planes is thus the system of the curve and torse. The curve is the edge of regression or cuspidal curve of the torse; in regard to the curve the points are points (or inounts), the lines tangents, and the planes osculating planes: in regard to the torse the points are points on the edge of regression, the lines generating lines, and the planes tangent planes.

2. Each line of the system is met by a certain number of non-consecutive lines, and the locus of the points of intersection (or say the locus of the intersection of two intersecting non-consecutive lines) is a nodal curve on the torse, or say simply it is the *nodal curve.* The plane containing the two intersecting non-consecutive lines envelopes a torse which is called the *nexal torse.*

3. There is occasion to consider

m, order of the system; this is the number of points of the system which lie in a given plane; or it is the order of the curve.

r, rank of the system; this is the number of lines of the system which meet a given line. It is thus the class of the curve; and the order of the torse.

n, class of the system; this is the number of planes of the system which pass through a given point. It is thus the class of the torse.

α, number of stationary planes; that is, planes each passing through four consecutive points of the system.

β, number of stationary points, that is, points each of them the intersection of four consecutive planes of the system.

g, number of *lines in two planes* (that is, lines each of them the intersection of two non-consecutive planes of the system) contained in a given plane; or say, number of apparent double planes of the torse.

G, number of double planes, or *tropes*, of the torse; viz. considering the torse as the envelope of a variable plane, if the plane in the course of its motion comes twice into the same position, we have then a double plane or trope.

h, number of *lines through two points* (that is, lines each through two non-consecutive points of the system) passing through a given point; or say, number of apparent double points of the curve.

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H, number of double points, or *nodes* of the curve; viz. considering the curve as described by a variable point, if the point in the course of its motion comes twice into the same position we have then a double point or node of the curve.

x, number of *points in two lines* (that is, points each of them the intersection of two non-consecutive lines of the system) contained in a given plane; or what is the same thing, order of nodal curve.

y, number of *planes through two lines* (that is, planes each containing two non-consecutive lines of the system) passing through a given point; or what is the same thing, class of the nexal torse.

ν, number of stationary lines of the system, that is, lines each containing three consecutive points of the system.

ω, number of double lines of the system, that is, lines each containing two pairs of consecutive points of the system.

t, number of *points on three lines* (that is, points each of them the common intersection of three non-consecutive lines of the system): these are also triple points on the curve.

γ, number of points of the system, through each of which passes a non-consecutive line of the system: these are intersections of the curve with the nodal curve, stationary points on the latter curve.

k, number of apparent double points of nodal curve.

q, class of nodal curve.

t', number of *planes through three lines* (that is, planes each of them through three non-consecutive lines of the system): these are also triple tangent planes of the torse.

γ', number of planes of the system each of them passing through a non-consecutive line of the system: these are common tangent planes of the torse and nexal torse, stationary planes of the latter torse.

k', number of apparent double planes of nexal torse.

q', order of nexal torse.

4. The formulæ thus contains in all the 21 quantities

$$m, r, n, \alpha, \beta, g, G, h, H, x, y, \nu, \omega \parallel t, \gamma, k, q \mid t', \gamma', k', q'.$$

My own Plückerian equations, or, say, the Plücker-Cayley equations, establish in all 6 relations between the first 13 quantities, and thus enable the expression of them in terms of any seven, say of

$$m, r, n, G, H, \nu, \omega,$$

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and the Salmon-Cremona equations then lead to the expressions in terms of these, of the remaining eight quantities $t, \gamma, k, q, t', \gamma', k', q'$.

I also consider

D_m, the deficiency of the curve,

D_x, the deficiency of the nodal curve.

I will first consider the equations themselves, and the mere algebraical transformations thereof; and afterwards the geometrical theory.

The Plücker-Cayley Equations.

5. These are found (as will be further explained) by considering first the cone, vertex an arbitrary point, which passes through the curve; and secondly, the section of the torse by an arbitrary plane. We have in the two figures

Cone.	*Section.*
m, order,	n, class,
r, class,	r, order,
$h + H$, double lines,	$g + G$, double tangents,
β, stationary lines,	α, stationary tangents,
$y + \omega$, double planes,	$x + \omega$, double points,
$n + \nu$, stationary planes.	$m + \nu$, stationary points.

And hence the two sets of quantities respectively are connected by the Plückerian relations, viz. these are

$$r = m(m-1) - 2\overline{h+H} - 3\beta,$$
$$n + \nu = 3m(m-2) - 6\overline{h+H} - 8\beta,$$
$$y + \omega = \tfrac{1}{2}m(m-2)(m^2-9) - (m^2-m-6)(2\overline{h+H} + 3\beta) + 2\overline{h+H}\,\overline{h+H-1} + 6h + H\beta + \tfrac{9}{2}\beta(\beta-1),$$
$$m = r(r-1) - 2(y+\omega) - 3(n+\nu),$$
$$\beta = 3r(r-2) - 6(y+\omega) - 8(n+\nu),$$

$$n = r(r-1) - 2(x+\omega) - 3(m+\nu),$$
$$\alpha = 3r(r-2) - 6(x+\omega) - 8(m+\nu),$$
$$G + G = \tfrac{1}{2}r(r-2)(r^2-9) - (r^2-r-6)(2\overline{x+\omega} + 3\overline{m+\nu}) + 2(x+\omega)(x+\omega-1) + 6(x+\omega)(m+\nu) + \tfrac{9}{2}(m+\nu)(m+\nu-1),$$
$$r = n(n-1) - 2(g+G) - 3\alpha,$$
$$m + \nu = 3n(n-2) - 6(g+G) - 8\alpha,$$

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$$h+H=\tfrac{1}{2}r(r-2)(r^2-9) \qquad x+\omega=\tfrac{1}{2}n(n-2)(n^2-9)$$
$$-(r^2-r-6)(3y-\omega+3n+v) \qquad -(n^2-n-6)(3g+G+3\alpha)$$
$$+2(y+\omega)(y+\omega-1) \qquad +2(g+G)(g+G-1)$$
$$+6(y+\omega)(n+v) \qquad +6(g+G)n$$
$$+\tfrac{1}{2}(n+v)(n+v-1), \qquad +\tfrac{3}{2}n(n-1),$$
$$n+v-\beta=3(r-n), \qquad n-(m+v)=3(n-r),$$
$$y+\omega-h-H=\tfrac{1}{2}(r-m)(r-n-9), \qquad g+G-x+\omega=\tfrac{1}{2}(n-r)(n+r-9),$$
$$\tfrac{1}{2}(r-1)(r-2)-(y+\omega)-(n+v), \qquad \tfrac{1}{2}(r-1)(r-2)-(x+\omega)-(m+v)$$
$$=\tfrac{1}{2}(n-1)(n-2) \qquad =\tfrac{1}{2}(n-1)(n-2)$$
$$-(h+H)-\beta. \qquad -(g+G)-\alpha,$$

and combining the two systems

$$\beta=\alpha+2(m-n),$$
$$y=x+(m-n),$$
$$h+H=g+G+\tfrac{1}{2}(m-n)(m+n-7),$$
$$\tfrac{1}{2}(m-1)(m-2)-(h+H)-\beta=\tfrac{1}{2}(n-1)(n-2)-(g+G)-\alpha,$$
$$\tfrac{1}{2}(r-1)(r-2)-(x+\omega)-(m+v)=\tfrac{1}{2}(r-1)(r-2)-(y+\omega)-(n+v).$$

6. Taking as data r, m, n, G, H, v, ω, we find very easily

$$h=\tfrac{1}{2}(m^2-10m-3n+8r-3v-2H),$$
$$g=\tfrac{1}{2}(n^2-10n-3m+8r-3v-2G),$$
$$x=\tfrac{1}{2}(r^2-r-n-3m-3v-2\omega),$$
$$y=\tfrac{1}{2}(y^2-r-m-3n-3v-2\omega),$$
$$\alpha=m-3r+3n+v,$$
$$\beta=n-3r+3m+v.$$

The Salmon-Cremona Equations.

7. These are

$$m(r-2)=2n+4\beta+\gamma+4v+4\omega+4H,$$
$$x(r-2)=n(r-4)+2\beta+3\gamma+3t+v(3r-14)+\omega(2r-10)+12H,$$
$$x(r-2)(r-3)=n(x-2r+8)+3mx+4k-3\alpha-9\beta-6\gamma$$
$$+v(3x-6r+18)+\omega(2x-4r+8)-12H,$$
$$q=x^2-x-2k-3\gamma-6t-3v(r-6)-2\omega(r-8)-2G-18H,$$
$$[=r(n-3)-3\alpha-2G],$$

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and

$$n(r-2) = 2m + 4\alpha + \gamma' + 4\nu + 4\omega + 4G,$$

$$y(r-2) = m(r-4)+2\alpha+3\gamma'+3t'+\nu(3r-14)+\omega(2r-10)+12G,$$

$$y(r-2)(r-3) = m(y-2r+8) + 3ny + 4k' - 3\beta - 9\alpha - 6\gamma' + \nu(3y-6r+18) + \omega(2y-4r+8) - 12G,$$

$$q' = y^2 - y - 2k' - 3\gamma' - 6t' - 3\nu(r-6) - 2\omega(r-8) - 2H - 18G,$$

$$[= r(m-3) - 3\beta - 2H],$$

where the second values of q, q' respectively are reduced forms obtained by the aid of the foregoing Plückerian relations.

8. Expressing these in terms of r, m, n, G, H, ν, ω, we obtain

$$\gamma = rn + 12r - 14m - 6n - 8\nu - 4\omega - 4H,$$

$$t = \tfrac{1}{6}[r^3 - 3r^2 - 58r - 3r(n+3m+3\nu+2\omega) + 42n + 78m + 78\nu + 48\omega],$$

$$k = \tfrac{1}{8}[r^4 - 6r^3 + 11r^2 + 66r - (2r^2 - 10r)(n+3m+2\nu+2\omega) + (n+3m+3\nu+2\omega)^2 - 58n - 126m - 126\nu - 76\omega - 24H],$$

$$q = rn + 6r - 3m - 9n - 3\nu - 2G,$$

and

$$\gamma' = rn + 12r - 14n - 6m - 8\nu - 4\omega - 4G,$$

$$t' = \tfrac{1}{6}[r^3 - 3r^2 - 58r - 3r(m+3n+3\nu+2\omega) + 42m + 78n + 78\nu + 48\omega],$$

$$k' = \tfrac{1}{8}[r^4 - 6r^3 + 11r^2 + 66r - (2r^2 - 10r)(m+3n+3\nu+2\omega) + (m+3n+3\nu+2\omega)^2 - 58m - 126n - 126\nu - 76\omega - 24G],$$

$$q' = rm + 6r - 3n - 9m - 3\nu - 2H.$$

9. We have thence

$$\gamma' - \gamma = -(r-8)(m-n) - 4(G-H),$$

$$t' - t = (r-6)(m-n),$$

$$q' - q = (r-6)(m-n) + 2(G-H),$$

$$k' - k = \tfrac{1}{2}(m-n)(r^2 - 5r - 2m - 2n - 3\nu - 2\omega + 17) - 3(G-H)$$

$$= \tfrac{1}{2}(m-n)(x+y-4r+17) - 3(G-H).$$

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10. Instead of obtaining the above values of γ, t, k, q directly it is convenient to verify them by substitution in the equations from which they were obtained; viz. writing for shortness $n+3m+3\upsilon+2\omega=P$, these may be written

$$-m(r-2)+2n+4\beta+4\upsilon+4\omega+4H+\gamma = 0,$$

$$-2x(r-2)+2r(P-3m)-8n+4\beta-28\upsilon-20\omega+6\gamma+24H+6t = 0,$$

$$-2x(r^2-5r+6-P)-4(r-4)n-18\beta-12\gamma-6\alpha+36\upsilon+16\omega-24H$$
$$-4r(-n-3m+P)+8k = 0,$$

$$-4q+2x(2x-2)-8k-8G-4r(-n-3m+P)+72\upsilon+64\omega$$
$$-12\gamma-24t-72H = 0,$$

which are to be satisfied by

$$\gamma = rm+12r-14m-6n-8\upsilon-4\omega-4H,$$

$$t = \tfrac{1}{6}[r^3-3r^2-58r-3rP+42n+78m+78\upsilon+48\omega],$$

$$k = \tfrac{1}{8}[r^4-6r^3+11r^2+66r-(2r^2-10r)P+P^2$$
$$-58n-126m-126\upsilon-76\omega-24H],$$

$$q = rn+6r-3m-9n-3\upsilon-2G,$$

$$x = \tfrac{1}{2}(r^2-r-P),$$

$$\alpha = m-3r+3n+\upsilon,$$

$$\beta = n-3r+3m+\upsilon,$$

$$P = n+3m+3\upsilon+2\omega.$$

11. We have, in fact, first

$$\left.\begin{array}{r} -mr+2m \\ +2n \\ +12m+4n-12r+4\upsilon \\ +4\upsilon \\ +4\omega \\ +4H \\ +mr-14m-6n+12r-8\upsilon-4\omega-4H \end{array}\right\} = 0,$$

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secondly

$$
\begin{array}{llll}
-r^3+ & r^2 & + rP & \\
+2r^2- & 2r & -2P & \\
 & +2rP & -6mr & \\
 & & - 8n & \\
 & -12r & + 4n+12m+ & 4v \\
 & & & -28v \\
 & & & -20\omega \\
 & +72r & +6mr-36n-84m- & 48v-24\omega-24H \\
 & & & +24H \\
+r^3-3r^2-58r-3rP & & +42n+78m+78v+48\omega & \\
\hline
\text{that is} & & -2P \quad +2n \ +6m \ +6v & = 0,
\end{array}
$$

thirdly

$$
\begin{array}{llll}
-r^4+5r^3- & 6r^2 & & \\
+ r^3- & 5r^2+ & 6r+P(2r^2-6r+6) & -P^2 \\
 & & -4rn+16n & \\
 & + 54r & -18n- & 54m- 18v \\
 & -144r & -12rm \quad +72n+168m+ & 96v+48\omega+48H \\
 & + 18r & -18n- & 6m- 6v \\
 & & & + 36v \\
 & & & +16\omega \\
 & & & -24H \\
 & & +P(-4r)+12rm+4rn & \\
+r^4-6r^3+11r^2+ & 66r+P(-4r^2+10r) & +P^2 & -58n-126m-125v-76\omega-24H \\
\hline
\text{that is} & & 6P \quad - 6n- 18m- & 18v-12\omega-24H = 0,
\end{array}
$$

and lastly

$$
\begin{array}{llll}
 & -24r & -4rn & +12m+36n+12v \quad +8G \\
+r^4-2r^3- & r^2+ 2r+P(-2r^2+2r+2)+P^2 & & +126m+58n+126v+76\omega+24H \\
-r^4+6r^3-11r^2- & 66r+P(2r^2-10r)-P^2 & & -8G \\
 & +P(-4r) & +4rn+12rm & +72v+64\omega \\
 & -144r & -12rm+168m+ & 72n+96v+48\omega+48H \\
-4r^3+12r^2+282r+P(& 12r) & & -812m-168n-812v-192\omega-72H \\
\hline
\text{that is} & & 2P \quad - 6m- 2n- & 6v- 4\omega \quad = 0,
\end{array}
$$

which completes the verification.

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12. The deficiency of the curve m is given by the equation

$$2D_m = r - 2m + 2 + \beta,$$

or substituting for β its value $= n - 3r + 3m + \nu$, this is

$$2D_m = m + n - 2r + \nu + 2.$$

The deficiency of the nodal curve x is given by

$$2D_x = q - 2x + 2 + \gamma + \nu(r-6) + 2H,$$

which, substituting for q, x, and γ, the values

$$rn + 6r - 3m - 9n - 3\nu - 2G,$$
$$\tfrac{1}{2}(r^2 - r - n - 3m - 3\nu - 2\omega),$$
$$rm + 12r - 14m - 6n - 8\nu - 4\omega - 4H,$$

respectively, becomes

$$2D_x = (r-14)(m+n) + \nu) - r^2 + 19r + 2 - 2\omega - 2G - 2H;$$

whence, also, writing herein $m+n+\nu = 2D_m + 2r - 2$, we have

$$2D_x - (r-14).2D_m = (r-5)(r-6) - 2\omega - 2G - 2H,$$

a relation between the two deficiencies.

Geometrical Theory of the foregoing Relations.

13. In considering the geometrical theory, we have to speak of the original curve, or curve of the system, and also of the nodal curve; it will be convenient to call them the curve m and the curve x respectively. I speak of the torse absolutely, to signify the torse of the system, as in what follows there is not the like occasion to speak of the nexal torse. I speak also of a plane α, meaning thereby any one of the stationary planes, the number of which is $= \alpha$; and so of a line α, meaning the line in the stationary plane α; and a point α, meaning the point of contact of such line with the curve m; or in the plural, the planes α, lines α, &c. And so in other cases; thus we have the stationary tangents ν, and the points ν, which are the points of contact hereof with the curve m. As regards a double tangent ω, we have here two points of contact; one of these separately would be a point ω; and we may speak of the points (or pair) 2ω, meaning thereby the two points of contact of the same tangent ω; or of the points 2ω, meaning the system of the 2ω points of contact of the tangents ω.

14. Observe that the expressions, the planes α, lines α, &c., have an absolute signification; there are other such ex-

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pressions which have only a relative signification, in regard to the system considered in connexion with a given point, line, or plane, as the case may require. Thus the expressions, *the lines g*, must be understood of the system in connexion with *a given plane*, to signify the lines in two planes contained in the *given plane;* the planes g, points g, would of course mean the planes or points of the system belonging to the lines g.

In particular the points m are the points of the system which lie in a given plane, the lines r are the lines which meet a given line; the planes n are the planes which pass through a given point. There will be occasion to speak, not only of the planes n, but also of the lines n and points n; these are of course the lines and points in the planes n.

15. It is to be remarked that, considering the torse as a surface, the nodal curve thereof is made up of the curve x and of the double lines ω (or its order is $= x + \omega$); the cuspidal curve is made up of the curve m, and of the stationary lines v (or its order is $= m + v$).

The Plücker-Cayley Equations.

16. The mode of obtaining these equations has already been indicated. We in fact consider the system in connexion with an arbitrary point, and with an arbitrary plane. The point is the vertex of a cone passing through the curve m, and this cone is of the order m, the class r, with $h + H$ double lines, β stationary lines, $y + \omega$ double tangent planes, and $n + v$ stationary tangent planes; viz. the order of the cone is equal to the number of lines in which this is intersected by a plane through the vertex; but each of these is determined as the line joining the vertex with an intersection of the plane by the curve m, and the number of them is thus $= m$. The class of the cone is equal to the number of the tangent planes which can be drawn through an arbitrary line through the vertex; but this is in fact the number of lines of the system which meet the arbitrary line, viz. it is $= r$. Again, any line drawn from the vertex to meet the curve twice, and also any line drawn to one of the points H, is a double line of the cone, that is, the whole number of double lines is $= h + H$. A line from the vertex to one of the points β is a stationary line of the cone; the number of these is $= \beta$. A plane through the vertex, and containing two tangents of the curve m, or containing a double tangent ω, is a double tangent plane of the cone, the

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number of these is thus $= y + \omega$. A plane through the vertex, which is also a plane of the system, is a stationary tangent plane; in fact, we have here on the curve m three consecutive points lying in a plane with the vertex, the tangent-plane of the cone is the plane through the vertex, and the first and second of the points on the curve; but this is also the plane through the vertex, and the second and third points, or the plane is a stationary tangent plane. But the plane through the vertex and the tangent v is also a stationary tangent plane of the cone; and the number of stationary tangent planes is thus $= n + v$.

17. Similarly for the section by the arbitrary plane; this is a curve of the order r and class n with $x + \omega$ double points, $m + v$ stationary points, $g + G$ double tangents, and α stationary tangents. In fact the order of the curve is equal to that of the torse; that is, to the number of lines which meet an arbitrary line, or $= r$. The class of the curve is equal to the number of tangents which pass through an arbitrary point of the plane; or, what is the same thing, the number of planes of the system which pass through this same arbitrary point, viz. it is $= n$. Each point of the plane which is the intersection of two lines of the system, and also each intersection of the plane by a line ω is a double point of the curve; viz. the number of these is $= x + \omega$. Each intersection with the curve m, and also each intersection with the tangent v, is a stationary point of the curve; viz. the number is $= m + v$. Each line in the plane, which is the intersection of two planes of the system, and also each intersection of the plane with a double tangent plane G, is a double tangent of the curve; viz. the number is $= g + G$. And, finally, each intersection of the plane with a stationary plane α is a stationary tangent of the curve; viz. the number of these is $= \alpha$.

18. We have thus the Plückerian relations as above between

$$m,\ r,\ h + H,\ \beta,\ y + \omega,\ n + v,$$

and between

$$n,\ r,\ g + G,\ \alpha,\ x + \omega,\ n + v.$$

Zeuthen's Tables.

18. The vertex of the cone and the plane of the section may occupy special positions. We have a table given by Zeuthen, but which I have completed so far as regards the double lines ω as follows:

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CONE THROUGH THE CURVE.

	Vertex.	*Order.*	*Class.*	*Double lines.*	*Stationary lines.*	*Double tangent planes.*	*Stationary tangent planes.*
1	Arbitrary	m	r	h $+H$	β	y $+\omega$	n $+\upsilon$
2	On a tangent	m	$r-1$	$h-1$ $+H$	β $+1$	$y-r+4$ $+\omega$	$n-2$ $+\upsilon$
3	On the curve	$m-1$	$r-2$	$h-m+2$ $+H$	β	$y-2r+8$ $+\omega$	$n-3$ $+\upsilon$
4	At a point H	$m-2$	$r-4$	$h-2m+6$ $+H-1$	β	$y-4r+20$ $+\omega$	$n-6$ $+\upsilon$
5	At a point β	$m-2$	$r-3$	$h-2m+6$ $+H$	$\beta-1$	$y-3r+13$ $+\omega$	$n-4$ $+\upsilon$
6	On stationary tangent υ	m	$r-2$	$h-2$ $+H$	β $+2$	$y-2r+9$ $+\omega$	$n-3$ $+\upsilon-1$
7	At point of contact of ditto	$m-1$	$r-3$	$h-m+1$ $+H$	β $+1$	$y-3r+14$ $+\omega$	$n-4$ $+\upsilon-1$
8	On double tangent ω	m	$r-2$	$h-2$ $+H$	β $+2$	$y-2r+10$ $+\omega-1$	$n-4$ $+\upsilon$
9	At point of contact of ditto	$m-1$	$r-3$	$h-m+1$ $+H$	β $+1$	$y-3r+15$ $+\omega-1$	$n-5$ $+\upsilon$

PLANE SECTION OF THE TORSE.

	Plane.	*Class.*	*Order.*	*Double tangents.*	*Stationary tangents.*	*Double points.*	*Stationary points.*
1	Arbitrary	n	r	g $+G$	α	x $+\omega$	m $+\upsilon$
2	Through a tangent	n	$r-1$	$g-1$ $+G$	α $+1$	$x-r+4$ $+\omega$	$m-2$ $+\upsilon$
3	A tangent plane	$n-1$	$r-2$	$g-n+2$ $+G$	α	$x-2r+8$ $+\omega$	$m-3$ $+\upsilon$
4	A double tangent plane G	$n-2$	$r-4$	$g-2n+6$ $+G-1$	α	$x-4r+20$ $+\omega$	$m-6$ $+\upsilon$
5	A stationary tangent plane α	$n-2$	$r-3$	$g-2n+6$ $+G$	$\alpha-1$	$x-3r+13$ $+\omega$	$m-4$ $+\upsilon$
6	Through stationary tangent υ	n	$r-2$	$g-2$ $+G$	α $+2$	$x-2r+9$ $+\omega$	$m-3$ $+\upsilon-1$
7	Tangent plane at contact of ditto	$n-1$	$r-3$	$g-n+1$ $+G$	α $+1$	$x-3r+14$ $+\omega$	$m-4$ $+\upsilon-1$
8	Through double tangent ω	n	$r-2$	$g-2$ $+G$	α $+2$	$x-2r+10$ $+\omega-1$	$m-4$ $+\upsilon$
9	A tangent plane at one of contacts of ditto	$n-1$	$r-3$	$g-n+1$ $+G$	α $+1$	$x-3r+15$ $+\omega-1$	$m-5$ $+\upsilon$

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19. To avoid confusion with the geometrical term line, I will speak (not of the lines, but) of the cases of these tables; the numbers in each case satisfy the foregoing Plückerian relations. To fix the ideas, I attend to the second table. We require to know in each case, say the numbers in the (n, r, α) columns; these being known, the other three numbers will be determined.

20. First for the n column; for the cases 2, 6, 8, the plane is not a tangent plane; the number of tangent planes which pass through a fixed point in the plane is still $= n$. For cases 3, 7, 9, the plane is a tangent-plane, it therefore counts 1 among the tangent-planes which pass through a fixed point thereof; and the number of the remaining tangent-planes is $= n - 1$. And so for cases 4 and 5, the plane counts for 2 among the tangent-planes which pass through a fixed point thereof, and the number of the remaining tangent-planes is $= n - 2$.

21. Next as to the r and α columns.

Case (2). The plane passes through a generating line, and therefore besides cuts the torse in a curve of the order $r - 1$; this generating line is a stationary tangent cutting the curve $r - 1$ in the point of contact with the curve m, counting 3 times (in all $\alpha + 1$ stationary tangents), and in $r - 4$ other points.

Case (3). The plane cuts the torse in the generating line counting twice, and in a curve of the order $r - 2$; the generating line is in regard to this curve an ordinary tangent at the point of contact with the curve m (so that the number of stationary tangents remains $= \alpha$), and besides cuts the curve in $r - 4$ points as in Case (2).

Case (4). The plane cuts the torse in two generating lines, each counting twice and in a curve of the order $r - 4$; each of the generating lines is in regard to this curve an ordinary tangent at the point of contact with the curve m (number of stationary tangents remains $= \alpha$), and besides cuts the curve in $r - 6$ other points.

Case (5). The plane cuts the torse in a generating line counting 3 times, and in a curve of the order $r - 3$; the generating line is in regard to this curve an ordinary tangent at the point of contact with the curve m, and besides cuts it in $r - 5$ points. The plane being in the present case a plane

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α, its intersections with the remaining planes α, gives the $\alpha - 1$ stationary tangents.

Case (6). The plane meets the torse in a generating line counting twice, and in a curve of the order $r-2$; the generating line is in regard to the curve a singular tangent meeting it in the point of contact with the curve m, counting 4 times, and besides meeting it in $(r-6)$ points. The generating line in respect of this four-pointic intersection counts as a stationary tangent twice; and the number of stationary tangents is $= \alpha + 2$.

Case (7). The plane meets the torse in the generating line counting 3 times, and in a curve of the order $r-3$; the generating line is in regard to this curve a stationary tangent at the point of contact with the curve m, counting 3 times, and besides meeting it in $r-6$ points as in Case 6. The whole number of stationary tangents is thus $= \alpha + 1$.

Case (8). The plane meets the torse in a generating line counting twice and in a curve of the order $r-2$. The generating line is in respect of the curve a stationary tangent at each of the points of contact with the curve m, viz. each of these points counts 3 times, and there are besides $r-8$ intersections. Moreover, the generating line counting as 2 stationary tangents, the whole number of stationary tangents is $= \alpha + 2$.

Case (9). The plane meets the torse in a generating line counting 3 times, and in a curve of the order $r-3$. The generating line is in respect of the curve an ordinary tangent at one of the points of contact with the curve m (viz. the point at which the plane is a tangent plane of the torse), so that we have here two intersections; and it is a stationary tangent at the other of the points of contact with the curve m (viz. the point at which the plane is not a tangent plane of the torse), so that there are here 3 intersections; there are therefore $(r-8)$ other intersections. The generating line reckons once as a stationary tangent, and the whole number of stationary tangents is $= \alpha + 1$.

22. The $\overline{r-6}$ points in Cases 6 and 7, are the points of intersection of the stationary tangent υ by other tangents of the curve m; and so the $\overline{r-8}$ points in Cases 8 and 9 are the points of intersection of the double tangent ω by other tangents of the curve m; these numbers $\overline{r-6}$ and $\overline{r-8}$ will present themselves in the sequel.

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We may as to the α-column sum up by saying, that in cases 2, 7, 9 there is a generating line which reckons as a stationary tangent; in Case 6 a generating line, which in respect of 4 consecutive intersections, reckons as two stationary tangents; and in Case 8 a generating line, which in respect of two pairs of 3 consecutive intersections, reckons as two stationary tangents.

In the $\overline{x+\omega}$ and $\overline{m+\nu}$-columns, observe that in cases (6), (7), (8), (9), we have in the first two ω, $\nu-1$, and in the last two $\omega-1$, ν; viz. these numbers refer to the intersections of the plane with the tangent ω, ν respectively; the actual numbers $\begin{pmatrix} x-2r+9 \\ +\omega \end{pmatrix}$ and $\begin{pmatrix} x-2r+10 \\ +\omega-1 \end{pmatrix}$, &c. are equal for the cases 6 and 8, and also for the cases 7 and 9. So in the $g+G$ column in case 4, we have $G-1$ for G.

The Nodal Curve x; Intersections with the Curve m; and Singularities.

23. The intersections of the curve m with the nodal curve x, are points α, β, γ, H, ν or ω.

24. At a point α, four consecutive points of the curve m lie in a plane; the point may be considered as the intersection of two consecutive tangents, viz. of the line through two consecutive points with that through the next two consecutive points; and it is thus a point on the curve x. We may imagine the points A, A' starting from α in opposite directions along the curve m, and moving in such manner that the tangents at these two points respectively continually intersect; we have thus a portion of the curve x, proceeding apparently in one direction only from the point α; and being, as regards the portion in question, an intersection of two real sheets of the torse; that is, a crunodal curve. The curve x, however, really extends in the opposite direction from α, but it is as to this portion thereof an intersection of two imaginary sheets of the torse; that is, an acnodal curve. The nodal curve x thus meets the curve m in the several points α, the curve x, each time that it passes through such a point of intersection, changing its character from crunodal to acnodal. The two *half sheets** of the torse cross each other

* In a curve (plane or twisted), the portions extending each way from a cusp (and considered without reference to a termination) are called half-branches; and so in a surface which has a cuspidal curve, or in particular, in a torse, the portions extending each way from the

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in the crunodal portion, extending in one direction from the point α of the nodal curve; and the acnodal portion extends in the opposite direction from the same point.

25. At a point β, or stationary point (cusp) on the curve m, four consecutive osculating planes meet at this point; and it is thus a point of intersection of two consecutive tangents (viz. of the line of intersection of two consecutive osculating planes, and of that of the next two consecutive osculating planes), consequently a point on the curve x. We may imagine the points B, B' starting from β, and moving along the two half-branches of the curve m, in such manner that the tangents at the two points respectively continually intersect. We have thus a portion of the curve x, proceeding apparently in one direction only from the point β (and having at this point a common tangent with the curve m), and being as regards this portion thereof an intersection of two real sheets of the torse; that is, a crunodal curve. The curve x, however, really extends in the opposite direction from the point β, but it is as to this portion thereof an intersection of two imaginary sheets of the torse; that is, an acnodal curve. The nodal curve thus meets the curve m in each of the points β, the curve x, each time that it passes through such a point, changing its character from crunodal to acnodal. There are at β three partial sheets of the torse; viz. if we imagine at this point the half-tangent b in the sense of the two half-branches of the curve m, and the half-tangent b' in the opposite sense, then we have through b' and one of the half-branches a partial sheet, and through b' and the other half-branch a partial sheet; these two partial sheets touching along b', and intersecting in the crunodal portion of the curve x; and a third partial sheet through b and the two half-branches of the curve m.

26. At a point γ on the cuspidal curve m, we have, traversing the curve and the two half-sheets which meet along it, another sheet of the torse, meeting the two half-sheets respectively in two half-branches, which are a portion of the nodal curve x, and which unite together (as at a cusp), in the point γ, which is thus a cusp or stationary point on the curve x.

cuspidal curve (and considered without reference to a termination) are called half-sheets. In the case of higher singularities of a like nature, we may speak of a partial branch or partial sheet (as the case may be).

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27. A point H is the intersection of two branches of the cuspidal curve m. There are for each branch two half-sheets; and we have thus at the point H four (say) quarter-branches of the curve x, touching each other at the point (viz. the common tangent is the intersection of the osculating planes belonging to the two branches of the curve m respectively). I find in a special manner that in regard to the curve x a point H is equivalent to six double points, plus two stationary points; it thus causes a reduction $2\cdot 6 + 3\cdot 2 = 18$ in the class of the curve.

28. The nodal curve x has in each of the double tangent planes G an actual double point. In fact the plane is an osculating plane of the curve m at two points thereof; that is, it contains two consecutive tangents R, R', and two other consecutive tangents S, S'; hence RS is a point on the nodal curve x; and not only so, but this is an actual double point, the two tangents being R and S; for since R is met by S and S', there is a consecutive point on the line R; that is, R is a tangent; and similarly since S is met by R and R' there is a consecutive point on the line S; that is, S is also a tangent.

29. At a point υ the tangent has with the curve x a 3-pointic intersection (whence also the tangent is a stationary tangent υ in regard to the curve x), the curves m and x have also a 3-pointic intersection at υ.

30. At each of the two points ω the tangent has with the curve x a 3-pointic intersection (viz. the tangent is in regard to the curve x more than a double tangent ω, instead of two 2-pointic intersections, or ordinary contacts, there are two 3-pointic intersections; I am unable to perceive this directly, but accept it on other grounds). But as at each of the points ω the intersection of the tangent with the curve m is 2-pointic, the intersection of the curves x and m is only 2-pointic.

31. The curves m and x meet in the points α each once, β each 3 times, γ each twice, υ each 3 times, 2ω each twice, and H each 8 times: we have thus

$$\alpha + 3\beta + 2\gamma + 3\upsilon + 4\omega + 8H$$

for the number of actual intersections of the two curves; and the number of apparent intersections is therefore

$$= mx - \alpha - 3\beta - 2\gamma - 3\upsilon - 4\omega - 8H,$$

a result which is required in the sequel.

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32. Consider the cone vertex an arbitrary point, through the curve x, or say simply the cone x.

Each line n (*quà* ordinary line of the system) is met by $r-4$ other lines, the $r-4$ points being situate on the curve x; the line n at each of these points *touches* the cone x, and it therefore besides intersects it in $(x-2r+8)$ points.

33. A line v meets the curve x in the point v counting 3 times, and in $r-6$ points each a stationary point of the curve; it consequently meets the cone x, in the point v counting 3 times, in each of the $(r-6)$ points counting twice, and besides in $x-2r+9$ points.

34. A line ω meets the curve x in the two points ω each counting 3 times, and in $r-8$ points each an actual double point of the curve; it consequently meets the cone x in the two points ω each counting 3 times, in the $(r-8)$ points each counting twice, and besides in $x-2r+10$ points.

35. Each of the $r-8$ points in which the double tangent ω meets another tangent of m is an actual double point of the curve x; and each of the $r-6$ points in which a stationary tangent v meets another tangent of m is a stationary point or cusp on the curve x. We have thus as singularities of the curve x, the k apparent double points, G actual double points, $\omega(r-8)$ ditto, t triple points, H 4-branch cuspidal points, γ stationary or cuspidal points, $v(r-6)$ ditto. In regard to the effect upon the class of the curve x, each of the points t is equivalent to 3 double points, and produces a reduction $=6$; each of the points H is equivalent to 6 double points $+2$ cusps, and produces a reduction $2\cdot6+3\cdot2$, $=18$ as already mentioned. We have thus the relation

$$q=x(x-1)-2k-2G-3v(r-6)-2\omega(r-8)-3\gamma-6t-18H,$$

which is one of the Salmon-Cremona equations.

36. The stationary points of the curve x are the points γ, the $v(r-6)$ points, and the points H, each counting as 2 stationary points; that is, we have

$$2D_x=q-2x+2+\gamma+v(r-6)+2H,$$

used above for finding the value of the deficiency D_x.

The remaining Salmon-Cremona Equations.

37. The formulæ for $m(r-2)$, $x(r-2)$ and $x(r-2)(r-3)$ correspond to those for $c(n-2)$, $b(n-2)$ and $b(n-2)(n-3)$

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in the case of a general surface (Salmon, *Solid Geometry*, 2nd Ed., Nos. 522 and 525), being obtained as follows; considering as before the cone, vertex an arbitrary point, which passes through the curve m, the total number of lines hereof which have with the torse a 3-pointic intersection at the curve m is $= m(r-2)$; and, similarly, considering the cone, vertex the same arbitrary point, which passes through the curve x, the total number of lines hereof which have with the torse a 3-pointic intersection at the curve x is $= x(r-2)$; viz. these numbers $m(r-2)$ and $x(r-2)$ are the numbers of the intersections of the two curves respectively with the surface of the order $(r-2)$, which is the second polar of the arbitrary point in regard to the torse. And so also in the cone, vertex the arbitrary point, which passes through the curve x, the total number of lines which touch the torse at a point not on the curve x is $= x(r-2)(r-3)$; viz. this is obtained by Salmon, the number of intersections of the curve x with a certain surface of the order $(r-2)(r-3)$ (Salmon, Nos. 269 and 273, writing r for n), but in a preferable manner by Cremona thus; the cone through the curve x meets the torse in the curve x counting twice and in a residual curve of the order $x(r-2)$; the lines in question are the tangents from the vertex to this curve, which is a curve meeting each line of the cone $r-2$ times; we may on the surface of the cone metrically, as in the plane, construct the polar of the vertex in regard to the curve of the order $x(r-2)$; viz. we have thus on the cone a curve meeting each generating line $(r-3)$ times; and this polar curve meets the curve of the order $x(r-2)$ in $x(r-2)(r-3)$ points (this would be clearly the case if only the curve of the order $xr-2$ were the complete intersection of the cone by a surface of the order $r-2$, for then the polar curve would be the intersection of the cone by the polar surface of the order $r-3$; but in the case in hand, where this is not so, some additional considerations would be required in order to sustain the result), and we have thus the number $x(r-2)(r-3)$ of the lines in question.

38. I consider the second polar surface, order $r-2$, which belongs to a given arbitrary point. Any point on the torse, such that the line joining it with the arbitrary point cuts the torse 3-pointically at the point on the torse, is a point on the second polar $r-2$. Such points are, as will be shewn, the points n, $n(r-4)$, β, γ, υ, 2ω, $\upsilon(r-6)$, $\omega(r-8)$, t, H.

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39. We may consider through any such point, and the arbitrary point either a particular section of the torse or any section whatever. If the line joining the two points has at the point in question a 3-pointic intersection with the section, then it has a 3-pointic intersection with the torse, viz. the point possesses the required property. For the points n and $n(r-8)$, the plane may be taken to be the osculating plane of the curve m. This meets the torse in the line n twice, and in a curve of the order $r-2$ touching this line at the point n and meeting it in the $r-4$ points. Hence, considering the complete section made up of the line twice and the curve of the order $r-2$, the line from the arbitrary point to the point n or to any one of the $r-4$ points meets the section 3-pointically; viz. the lines to the points n and to the points $n(r-4)$ meet the torse 3-pointically.

40. For the points β; we may consider any section through the arbitrary point and β; in effect any section through the point β. A plane section through a point β has at this point an invisible triple point, that is a point not in appearance differing from an ordinary point of the curve, but which by considering a consecutive position of the plane of section is seen to be equivalent to a double point and two cusps; viz. the node is a point of intersection of the plane with the curve x, the cusps are the two intersections with the curve m, in the neighbourhood of the point β. The line from β to the arbitrary point has thus with the torse a 3-pointic intersection at β.

41. Similarly for the points γ; we take any section through the arbitrary point and γ; in effect any section through the point γ. The section through a point γ has at this point a triple point, at which an ordinary branch passes through a cusp, and thus equivalent to a cusp + 2 nodes; in fact, for a consecutive position of the cutting plane, the section has actually a cusp and two nodes; the cusp at the intersection of the plane with the curve m, the nodes at the two intersections of the plane with the curve x in the neighbourhood of the cusp γ. The line through the point γ has thus a 3-pointic intersection with the torse.

42. For a point v I consider the section through the tangent v; this is made up of the tangent twice, and of a curve of the order $r-2$ having with the tangent a 4-pointic intersection at the point v, and besides meeting it in $(r-6)$ points. Hence in the plane, a line through v, or through one of the

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$(r-6)$ points has at such point a 3-pointic intersection with the curve. And thus the lines through the points v and $v(r-6)$ respectively have a 3-pointic intersection with the torse.

43. Similarly for a point ω, I consider the section through a tangent ω; this is made up of the tangent twice, and of a curve of the order $r-2$ having with the tangent a 3-pointic intersection at each of the points ω, and besides meeting it in $r-8$ points. Hence in the plane a line through either of the points ω or through one of the $r-8$ points has with the curve a 3-pointic intersection, and thus the lines through the points 2ω and $\omega(r-8)$ respectively have a 3-pointic intersection with the torse.

44. For a point H I consider any section through the arbitrary point and H; in effect any section through H. There is at H a singularity $= 6$ nodes $+ 2$ cusps. But a line through H in the plane of the section cuts the section in 4 points only; that is, the line from the arbitrary point to H has with the curve a 4-pointic intersection at H; and *à fortiori* it may be regarded as a line of 3-pointic intersection. The lines to the points H have thus (a 4-pointic, that is, more than) 3-pointic intersection with the torse.

45. For a point t I consider any section through the arbitrary point and t; in effect any section through t; there is at t a triple point, and the line through t has thus a 3-pointic intersection at t. Hence a line through t has 3-pointic intersections with the torse.

46. The several points above referred to lie on the curve m or on the curve x, or on both of these curves; and each curve has at these points respectively a simple or multiple intersection with the polar surface $r-2$, as shewn in the table.

		Intersection of second polar $r-2$	
		with curve m.	*curve x.*
Points	n	2	0
,,	$n(r-4)$	0	1
,,	β	4	2
,,	γ	1	3
,,	v	4	4
,,	2ω	2	3
,,	$v(r-6)$	0	3
,,	$\omega(r-8)$	0	2
,,	H	4	12
,,	t	0	3

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where the figures 1, 2 &c. denote a simple intersection, 2-pointic intersection &c. of the curve and surface; 0 denotes of course that there is not any intersection, viz. that the curve does not pass through the point referred to.

47. Several of the foregoing numbers are obtained without difficulty; thus we see that the points n, $n(r-4)$ are ordinary points on the second polar $r-2$, the surface at each of the points n *touching* the curve m, but at each of the points $n(r-4)$ simply cutting the curve x. So also the points β, γ, υ, 2ω, $\upsilon(r-6)$, $\omega(r-8)$ are ordinary points on the second polar; at a point β the two half-branches of the curve m touch the surface in a special manner so as to give a 4-pointic intersection; whereas the curve x simply touches the surface. At γ the curve m cuts the surface, but the two half-branches of x *touch* the surface. At υ, each of the curves m, x has a 4-pointic intersection with the surface; at each of the two points ω, the curve m touches the surface, but the curve x has with it a 3-pointic intersection, but at $\omega(r-8)$ it simply touches the surface.

48. The point H is in the nature of a biplanar point on the polar surface; this appears, or is at least indicated, by the circumstance that the line to the arbitrary point has with the torse not a 3-pointic but a 4-pointic intersection; the two branches of the curve m each simply cut the two coincident sheets of the polar surface, giving $2 \times 2, = 4$ intersections; but for the curve x, the four partial branches each touch the two coincident sheets; for a single sheet the number of intersections would be 6, but for the two coincident sheets it is twice this or $= 12$. Finally, a point t is an ordinary point on the second polar, each of the three branches of x simply cuts the surface, or the number of intersections is $= 3$.

49. The last table gives at once

$$m(r-2) = 2n + 4\beta + \gamma + 4\upsilon + 2.2\omega + 4H,$$

$$x(r-2) = n(r-4) + 2\beta + 3\gamma + 4\upsilon + 3.2\omega + 3\upsilon(r-6) + 2\omega(r-8) + 12H + 3t,$$

which are the true theoretical forms of the equations for $m(r-2)$ and $x(r-2)$, in which these were obtained by Cremona.

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50. The $x(r-2)(r-3)$ points are those points in which the Cremona $x(r-2)$ curve is met 2-pointically by the line from the arbitrary point (I recall that taking the arbitrary point as the vertex of a cone through the curve x, this cone, say the cone x, meets the torse in the curve x twice, and in the $x(r-2)$ curve in question); viz. these points are either points of contact of tangents from the vertex to the $x(r-2)$ curve; or they are double points, or else cusps of the $x(r-2)$ curve; in which several cases respectively they count 1, 2 or 3 times, among the $x(r-2)(r-3)$ points.

51. The points of contact are the $n(x-2r+8)$ points of intersection of the lines n with the cone x. We have in fact a plane n through the vertex of the cone, and in this plane two consecutive lines of the system; hence at each of the $x-2r+8$ points the generating line of the cone meets the two consecutive lines of the system; that is, there is with the curve $x(r-2)$ a 2-pointic intersection, not arising out of any singularity of the curve, and consequently a contact of this curve with the generating line of the cone.

52. The actual double points of the curve $x(r-2)$ are first the $2k$ apparently coincident points of the curve x, and secondly the $\omega(x-2r+10)$ points on the lines ω. For first if we consider through the vertex a line meeting the curve x in two points, say A, B, this meets the torse in these points each twice and in $r-4$ other points. Now imagine a line from the vertex to the point P in the vicinity of A, this meets the torse in the point P twice and in $r-2$ points, which are points on the $x(r-2)$ curve; hence as P travels through A, 2 of the $r-2$ points come together at B, and again separate, that is B is an actual double point on the $x(r-2)$ curve; and similarly A is an actual double point on the curve; and we have thus the $2k$ double points. Secondly, since the line ω is a nodal line on the torse, a generating line of the cone, in the neighbourhood of and considered as travelling through one of the $x-2r+10$ points, meets the torse in two points which come to concide and then again separate; this is each of the $x-2r+10$ points is an actual double point on the curve $x(r-2)$; and the whole number of these is $=\omega(x-2r+10)$.

53. The stationary points of the curve $x(r-2)$ are first the points on the curve m which apparently coincide with the curve x; viz. the number of these, as was seen, is

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$= mx - \alpha - 3\beta - 2\gamma - 3\nu - 4\omega - 8H$; secondly, the $\nu(x - 2r + 9)$ points on the lines ν; thirdly, the points H each counting as 4 cusps. For first consider a generating line meeting the curve x in B and the curve x in A; if we imagine on the curve x a point Q which approaches and ultimately coincides with B, the generating line through Q meets the torse in the neighbourhood of its cuspidal edge in two points which come ultimately to coincide with the point A, and we thus see that A is a stationary point on the $x(r-2)$ curve.

54. Secondly, observing that the line ν is a cuspidal line on the torse, and considering in like manner a generating line of the x cone, which approaches and comes ultimately to coincide with one of the $x - 2r + 9$ points, we see that this is a stationary point on the $x(r-2)$ curve. And thirdly, any line through a point H meets the torse in this point counting 4 times, and in $r-4$ other points. Hence considering the generating line of the x cone, which travelling along any one of the four partial branches of the x curve comes ultimately to coincide with H, 2 of the $r-2$ points on such generating line come to coincide at the point H; and we have thus the point H as a singular point on the $x(r-2)$ curve; viz. it reckons as a stationary point once in respect of each of the four partial branches of the curve x (it must be assumed that this is so, but a further proof is required), that is as 4 cusps on the $x(r-2)$ curve.

55. By what precedes we have

$$x(r-2)(r-3) =$$
$$n(x - 2r + 8)$$
$$+ 2\{2k + \omega(x - 2r + 10)\}$$
$$+ 3\{(mx - \alpha - 3\beta - 2\gamma - 3\nu - 4\omega - 8H) + \nu(x - 2r + 9) + 4H\},$$

which is the true theoretical form in which the equation for $x(r-2)(r-3)$ was obtained by Cremona.

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DEFORMATION OF AN ELASTIC SPHERE PRESSED BETWEEN TWO PARALLEL PLANES.

By R. Hoppe.

In the Philosophical Transactions of the *Royal Society of London*, Vol. 153, Mr. (now Sir) W. Thomson has resolved the general problem: A shell consisting of isotropic elastic material, and bounded by two concentric spherical surfaces being given, and displacements being arbitrarily produced or forces arbitrarily applied over its whole bounding surface, to find the displacement of every point of its substance.

It is obvious that the condition concerning the surface does not include every possible case, since it requires that the displacements or forces are applied to the whole surface. What occurs most frequently, is that a part of the surface is deformed in a given manner, and the effect experienced by every particle of the substance is to be determined. As the most simple case of this sort, I propose to find the displacements of all points of a solid sphere pressed between two parallel planes of a given distance, a little smaller than the diameter. The material is likewise supposed isotropic, the planes perfectly rigid and without friction. To realize the exact conditions, we may think an unlimited range of spheres, all of equal size, and pressed against each other; each of them will be in the state to be supposed here.

The given conditions for the surface are the following:

The normal and tangential strains are in general zero, except for two small areas around the poles, where the normal strains are unknown, and instead of them the two planes are given, into which all points are transferred.

It is, however, known beforehand that the displacement of any point is independent of its azimuth, and that any azimuth remains unvaried by the deformation. If, therefore, the original coordinates of an element, referring to the centre as origin and to the normal of the pressing planes as axis of x,

$$x = \kappa \cos\lambda,\ y = \kappa \sin\lambda,\ z,$$

receive by the deformation the increments ξ, η, ζ, we shall have

$$\xi = \kappa \cos\lambda,\ \eta = \kappa \sin\lambda,$$

and κ, ζ will not depend on λ.

Let now Θ denote the increment of any unity of volume, and m the ratio of the transversal and longitudinal elasticity,

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that means, that a cubical element longitudinally compressed is shortened m times as much as it is stretched in either transversal direction. Then the conditions of internal equilibrium are

$$\left.\begin{aligned} \frac{m}{m-2}\frac{d\Theta}{dx}+\frac{d^2\xi}{dx^2}+\frac{d^2\xi}{dy^2}+\frac{d^2\xi}{dz^2}=0,\\ \frac{m}{m-2}\frac{d\Theta}{dy}+\frac{d^2\eta}{dx^2}+\frac{d^2\eta}{dy^2}+\frac{d^2\eta}{dz^2}=0,\\ \frac{m}{m-2}\frac{d\Theta}{dz}+\frac{d^2\zeta}{dx^2}+\frac{d^2\zeta}{dy^2}+\frac{d^2\zeta}{dz^2}=0,\\ \Theta=\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}. \end{aligned}\right\} \quad \ldots\ldots (1).$$

The equilibrium on the surface requires (1), that the tangential strains are throughout zero, which is expressed by the equations

$$\left.\begin{aligned} 2x\left(L-\frac{d\xi}{dx}\right)=y\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right)+z\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right),\\ 2y\left(L-\frac{d\eta}{dy}\right)=z\left(\frac{d\eta}{dx}+\frac{d\zeta}{dy}\right)+x\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right),\\ 2z\left(L-\frac{d\zeta}{dz}\right)=x\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right)+y\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right), \end{aligned}\right\} \quad \ldots.(2),$$

(2) that the normal strain normal to the surface vanishes at the free surface, which is expressed by the equation

$$(m-2)L+\Theta=0;$$

(3) that at the flattened parts of the surface,

$$z+\zeta=\pm(c-\gamma),$$

where c is the radius of the sphere, $c-\gamma$ the half distance of the planes.

The equations for internal equilibrium may be satisfied by the values

$$\xi=a\frac{dV_n}{dx}+cr^2\frac{dV_{n-2}}{dx},$$

$$\eta=a\frac{dV_n}{dy}+cr^2\frac{dV_{n-2}}{dy},$$

$$\zeta=(a-b)\frac{dV_n}{dz}+cr^2\frac{dV_{n-2}}{dz},$$

$$\Theta=\{2(n-2)c-n(n-1)b\}V_{n-2},$$

$$r^2=x^2+y^2+z^2,$$

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where V_n is an homogeneous function of κ, and z satisfying the equations

$$\left.\begin{aligned} &\frac{d^2V_n}{dx^2}+\frac{d^2V_n}{dy^2}+\frac{d^2V_n}{dz^2}=0,\\ &\frac{dV_n}{dz}=nV_{n-1};\quad V_0=1,\\ &x\frac{dV_n}{dx}+\frac{dV_n}{dy}+z\frac{dV_n}{dz}=nV_n. \end{aligned}\right\}\ldots\ldots\ldots\ldots(3).$$

Besides, if we put

$$V_n=r^nv_n,$$

v_n will be a spherical function of one variable, completely determined by the recurrent relation

$$nv_{n-1}-(2n+1)\frac{z}{r}v_n+(n+1)v_{n+1}=0\ldots\ldots\ldots(4).$$

Substituting those values in (1), we find

$$n(n-1)b=\left\{2(3n-5)-2\frac{2n-3}{m}\right\}c^2,\ (n\gtrless 2),$$

$$\Theta=-2(2n-3)\frac{m-2}{m}cV_{n-2},\quad (n\gtrless 2),$$

$$\Theta=-2b,\qquad\qquad (n=2).$$

On account of the symmetry of the figure each of the components of displacement is to change sign with the corresponding coordinate, while the two other components as well as Θ remain unchanged. Hence, it follows that n can only have even values. Moreover, continuity at the centre requires that no negative power of r enters into the expressions, and for this reason 2 is the least value of n. Collecting together all particular solutions and passing to meridional coordinates κ, z, we obtain the following expressions:

$$\left.\begin{aligned} &\kappa=\sum_{n=2}^{n=\infty}\left\{a_n\frac{dV_n}{d\kappa}+c_nr^2\frac{dV_{n-2}}{d\kappa}\right\},\\ &\zeta=\sum_{n=2}^{n=\infty}\left\{(a_n-b_n)\frac{dV_n}{dz}+c_nr^2\frac{dV_{n-2}}{dz}\right\}, \end{aligned}\right\}\ldots\ldots\ldots(5),$$

$$\Theta=-2b_2-2\frac{m-2}{m}\sum_{n=4}^{n=\infty}(2n-3)c_nV_{n-2}\ldots\ldots\ldots(6),$$

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where a_n, b_n, c_n for odd values of n are zero, while all other a_n and c_n as well as b_2 remain arbitrary constants, and

$$b_n = \frac{2}{n} \frac{3n-5-2\frac{2n-3}{m}}{n-1} c_n, \quad (n>2) \ldots\ldots (7).$$

The superficial equations (2) reduced to meridional coordinates give

$$2kL = 2\frac{kd\kappa + zd\zeta}{dk} + z\left(\frac{d\kappa}{dz} - \frac{d\zeta}{dk}\right),$$

$$2zL = 2\frac{kd\kappa + zd\zeta}{dz} - k\left(\frac{d\kappa}{dz} - \frac{d\zeta}{dk}\right),$$

whence $$r^2 L = k\left(k\frac{d\kappa}{dk} + z\frac{d\kappa}{dz}\right) + z\left(k\frac{d\zeta}{dk} + z\frac{d\zeta}{dz}\right),$$

$$0 = 2kz\left(\frac{d\kappa}{dk} - \frac{d\zeta}{dz}\right) + (z^2 - k^2)\left(\frac{d\kappa}{dz} + \frac{d\zeta}{dk}\right).$$

Introducing the values (5) expressed by spherical functions, we find

$$L = \sum_{n=2}^{n=\infty} (n-1)\left\{n\left(a_n v_n - b_n \frac{z}{c} v_{n-1}\right) + (n-2)\, c_n v_{n-2}\right\} c^{n-2} \ldots (8),$$

$$0 = \sum_{n=2}^{n=\infty} (n-1)\left[2n\frac{n+1}{2n+1}\left(a_n - \frac{nb_n}{2n-1}\right)(v_{n+1} - v_{n-1}) - \frac{n-2}{2n-3}\left\{\frac{nb_n}{2n-1} - 2(n-2)\, c_n\right\}(v_{n-1} - v_{n-3})\right] c^n.$$

The latter equation admits of an immediate solution. Putting for brevity,

$$d_n = a_n - \frac{nb_n}{2n-1}; \quad e_n = \frac{b_n}{2n-1} - 2\frac{n-2}{n} c_n \ldots\ldots (9),$$

we may make each term of the first part of the series destroy the subsequent term of the second part, by establishing the relation

$$d_n = \frac{1}{2}\frac{n+2}{n-1} e_{n+2} c^2 \ldots\ldots\ldots\ldots (10).$$

The equations (7), (9), (10) are sufficient to reduce all constant coefficients to b_2, c_2, c_4, c_6, ... To simplify their expressions, let

$$F_n = \frac{(n-1)(2n-1)}{n^2 - 3n + 3 + \frac{2n-3}{m}};$$

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then we have

$$b_n = 2\frac{2n-1}{n}\left(1-\frac{2}{F_n}\right)c_n,\ (n>2) \ \ldots\ldots\ (11),$$

$$e_n = -\frac{2}{n}\left(n-3+\frac{2}{F_n}\right)c_n,\ (n>2),$$

$$d_n = -\left\{1+\frac{2}{(n-1)F_{n+2}}\right\}c_{n+2}c^2 \ \ldots\ldots\ (12),$$

$$a_n = 2\left(1-\frac{2}{F_n}\right)c_n - \left\{1+\frac{2}{(n-1)F_{n+2}}\right\}c_{n-2}c^2,\ (n>2),$$

$$a_2 = \tfrac{2}{3}b_2 - \left(1+\frac{2}{F_4}\right)c_4c^2.$$

Now, if E signifies longitudinal elasticity, and

$$N = L + \frac{\Theta}{m-2},$$

the normal pressure on unity of surface is

$$= \frac{m}{m+1}EN.$$

Substituting the values (8), (6), we find

$$N = -\frac{mb_2}{m-2} + c_2 + \sum_{n=-\infty}^{n=\infty}\left\{n(n-1)\left(a_nv_n + \frac{b_n}{2}\frac{v_{n-2}-2nv_n}{2n-1}\right) + (n^2-6n+7)c_nv_{n-2}\right\}c^{n-2},$$

or, according to (9),

$$N = -\frac{mb_2}{m-2} + c_2 + \sum_{n=2}^{n=\infty}\left[n(n-1)d_nv_n + \left\{\frac{n}{2}\frac{n-1}{2n-1}b_2 + (n^2-6n+7)c_n\right\}v_{n-2}\right]c^{n-2};$$

or, collecting the spherical functions of equal order,

$$N = -\frac{2}{3}\frac{m+1}{m-2}b_2 + \sum_{n=2}^{n=\infty}\left\{\frac{n(n-1)}{c^2}d_n + \frac{n+2}{2}\frac{n+1}{2n+3}b_{n+2} + (n^2-2n-1)c_{n+2}\right\}c^nv_n,$$

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and reducing the constants according to the formulæ (11) (12),

$$N=-\frac{2}{3}\frac{m+1}{m+2}b_2-2\sum_{n=2}^{n=\infty}\frac{2n+1}{F_{n+2}}c_{n+2}c^n v_n \ldots\ldots (13).$$

For any point of the spherical surface let

$$t=\frac{z}{c},$$

and let $t=h$ express the periphery of the area, which is flattened by the deformation: N will be $=0$ between $t=-h$ and $t=h$; whence

$$\int_{-1}^{1} Nv_n dt=\int_{-1}^{-h} Nv_n dt+\int_{h}^{1} Nv_n dt=2\int_{h}^{1} Nv_n dt.$$

Any term of the series $v_1, v_2, v_3, \ldots$ except v_n, being substituted for N, the integral on the left will vanish. Besides it is known, that

$$\int_{-1}^{1} v_n^2 dt=\frac{2}{2n+1}.$$

Introducing, therefore, the expressions (13), we obtain

$$\int_{-1}^{1} Nv_n dt=-\frac{4c_{n+2}}{F_{n+2}}e^n.$$

Thus it follows that

$$c_{n+2}=-\frac{F_{n+2}}{2c^n}\int_{h}^{1} Nv_n dt \ldots\ldots\ldots\ldots (14).$$

Likewise we find

$$b_2=-\frac{3}{2}\frac{m-2}{m+1}\int_{h}^{1} N dt.$$

Within the limits of these integers, $1-t$ is of the same order of smallness as the displacement of the poles $(1-h)c$. Therefore, according to the principles of the theory of elasticity we may consider N, which is a function of t, as a linear function of $1-t$. Because it is to vanish for $t=h$, it must have the form

$$N=A(t-h),$$

where A is an unknown constant.

To effect the integration in (14), we have the formula

$$(n+1)v_n dt=d(v_n t-v_{n-1}) \ldots\ldots\ldots\ldots (15),$$

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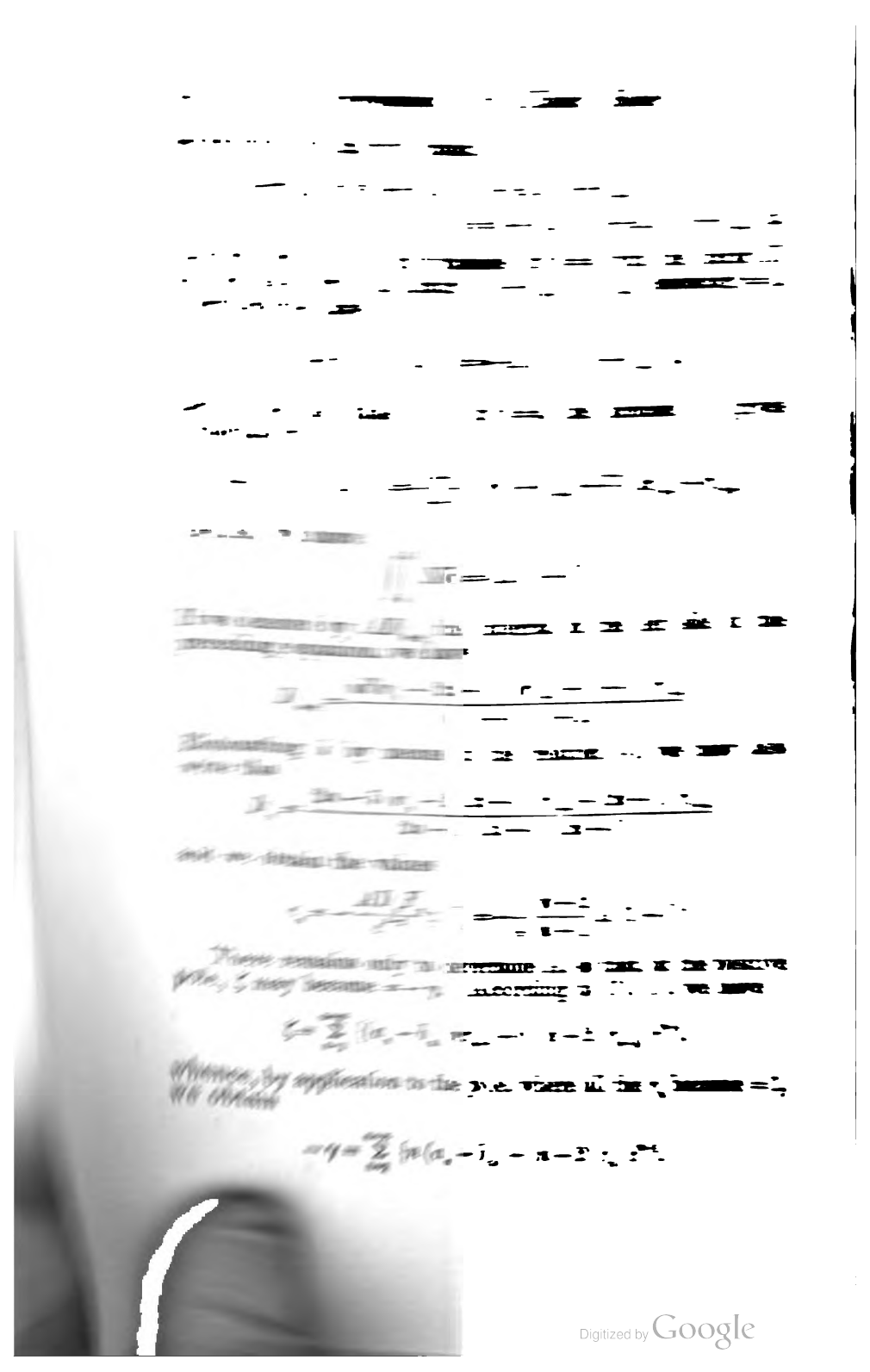

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Substituting the expression indicated for a_n, b_n, c_n, we find

$$-\gamma=\frac{1}{2}\frac{m-2}{m+1}Ac(1-h)^2+2Ac\sum_{n=4}^{n=\infty}\left\{(n-1)F_n-\frac{2n^2-9n+8}{n-3}\right\}D_n.$$

The separated term, as a quantity of the second order, needs not to be brought into account.

To collect together the results, let us put, for brevity,

$$G=-\sum_{n=2}^{n=\infty}\left\{(2n-1)F_{2n}-2\frac{4n^2-9n+4}{2n-3}\right\}D_{2n},$$

we shall have $$A=\frac{\gamma}{2cG},$$

$$a_n=\frac{\gamma}{Gc^{n-1}}\left\{(2-F_n)D_n+\left(\frac{2}{n-1}+F_{n+2}\right)D_{n+2}\right\},$$

$$b_n=\frac{\gamma}{Gc^{n-1}}\frac{2n-1}{n}(2-F_n)D_n,$$

$$c_n=-\frac{\gamma D_n}{2Gc^{n-1}},$$

applicable to even numbers $n>2$; and

$$a_2=-\frac{1}{4}\frac{m-2}{m+1}\frac{\gamma(1-h)^2}{Gc}+\frac{\gamma}{Gc}(2+F_4)D_4,$$

$$b_2=-\frac{3}{8}\frac{m-2}{m+1}\frac{\gamma(1-h)^2}{Gc}.$$

To determine h we ought to make $\zeta=-\gamma$ also for $t=h$, which would give a transcendent equation. But, since γ differs only in the second order from $(1-h)c$, it is permitted to put

$$h=1-\frac{\gamma}{c},$$

and to use this condition as a control.

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NOTE ON SIN ∞ AND COS ∞.

By WILLIAM WALTON, M.A., Fellow of Trinity Hall.

VARIOUS mathematical writers have concluded that the values of $\sin\infty$ and $\cos\infty$ are both zero; others have objected to such a view. The state of the argument is well exhibited in an interesting article by Mr. Glaisher of Trinity College in the fifth volume of the *Mathematical Messenger*, p. 232, where he has also shewn that the coexistence of the said paradoxical values of $\sin\infty$ and $\cos\infty$ requires the basis of certain definitions.

The question may be viewed also, as I propose to shew, under the light of a fundamental principle of indices. It is a proposition in the theory of indices, as ordinarily stated in text books, that a^0 is equal to 1. This proposition appears to me to be erroneous, if 0 be taken to represent infinitesimal magnitude. I should substitute the following proposition, viz. that "1 is the limiting value of a^n when n is indefinitely diminished." The limiting value of an expression cannot always, except in the final result of an operation, replace the expression itself.* In fact, 0 denoting infinitesimal magnitude, then, if $m = n + 0$,

$$\frac{a^m}{a^n} = a^{m-n} = a^0:$$

if $m = n$, then

$$1 = \frac{a^m}{a^n} = a^{m-m},$$

where $m - m$ stands for nonentity and where a^{m-m} is an expression without intrinsic signification. If a^0 were equal to 1, then 0, a quantity, should be equal to $m - m$, a nonentity. Such an expression as a^{m-m} may no doubt occur in algebraic researches, not as having signification in itself, but as standing in place of, what it should not strictly have displaced, viz. 1; just as in the expansion of $(a + x)^n$ by the Binomial theorem, the coefficient of the first term, as a case of the coefficient C_{r-1} of the r^{th} term, is C_0, a meaningless quantity, occupying the place of 1.

* For example, 0 denoting, in what follows, infinitesimal magnitude,

$$a\int_a^{\frac{1}{a^2}} e^{-x^2}dx = e^{-a^2} - e^{-\frac{1}{a}}$$

$$= 1, \text{ when } a = 0;$$

whereas

$$a\int_a^{\frac{1}{a^2}} dx = \frac{1}{a} - a^3 = \infty, \text{ when } a = 0.$$

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Take one of the formulæ, of which Mr. Glaisher and others have made use in connection with the present question, viz.

$$\int e^{-ax} \sin x\,dx = -\frac{e^{-ax}}{1+a^2}(\cos x + a \sin x) \ldots\ldots (1).$$

Take, in accordance with Mr. Glaisher's view, the limits a^2, $\frac{1}{a^2}$: then, finally replacing a by 0, an infinitesimal magnitude,

$$\int_{a^2}^{\frac{1}{a^2}} e^{-ax} \sin x\,dx = 1.$$

If, in the expression

$$\int_{a^2}^{\frac{1}{a^2}} e^{-ax} \sin x\,dx$$

we replace a by nonentity, and then replace the meaningless expression e^{-ax} by 1, we should have to integrate $\sin x\,dx$ from the square of nonentity to the reciprocal of the square of nonentity, that is, we should not be executing any real computation, but merely playing with algebraic symbols and arriving at results unconnected with logic.

Mr. Glaisher's definition of $\cos\infty$, viz.

$$\cos\infty = 1 - \int_{a^2}^{\frac{1}{a^2}} e^{-ax} \sin x\,dx \ldots\ldots\ldots\ldots (2),$$

where a denotes infinitesimal magnitude, does not I think, from what I have said, agree with the strict theory of indices, although I agree with him in his conclusion that it is a definition essential to the supposition that $\cos\infty = 0$. The left-hand member of the equation (2) is obtained by taking a to represent nonentity and integrating in form without the essential idea of integration; while on the right hand a is taken to denote infinitesimal magnitude.* The question as to $\sin\infty$ may be discussed in like manner.

Finally, it appears to me that, under every aspect, the formulæ $\sin\infty = 0$ and $\cos\infty = 0$ are grounded on no logical basis, since their demonstration can be effected only by pursuing algebraic processes which do not express ideas.

* The necessity of regarding a as a quantity on the right-hand side will be obvious from the investigation of the relation (1).

July 25, 1871.

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ON THE SUMMATION BY DEFINITE INTEGRALS OF GEOMETRICAL SERIES OF THE SECOND AND HIGHER ORDERS.

By J. W. L. GLAISHER, B.A., F.R.A.S.

THE summation of a series of terms, whose exponents form an arithmetical series of the second order, such as $q+q^4+q^9+q^{16}+\ldots$, which may conveniently be called a geometrical series of the second order, has been effected so as to exhibit the result in the form of a definite integral in several different ways, by Cauchy, Jacobi, Kummer, and Schlömilch.

All these analysts have obtained their summations by means of the integral $\int_0^\infty e^{-x^2}\cos 2ax\,dx=\frac{\sqrt{\pi}}{2}e^{-a^2}$, with the exception of Jacobi, who made use of the elliptic integrals in the well known results

$$1+q+q^4+q^9+q^{16}+\ldots=\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\left(\frac{2K}{\pi}\right)},$$

$$1-q+q^4-q^9+q^{16}-\ldots=\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\left(\frac{2k'K}{\pi}\right)}.$$

(*Fundamenta Nova*, p. 184).

The integrals

$$\int_0^\infty \frac{a\cos bx}{a^2+x^2}\,dx=\tfrac{1}{2}\pi e^{-ab},\text{ and }\int_0^\infty \frac{x\sin bx}{a^2+x^2}\,dx=\tfrac{1}{2}\pi e^{-ab}$$

are however also available for the purpose; and in some respects, admit of more general application to this class of series, as they afford a means of expressing the sums of a geometrical series of any order; but the results are so complicated as to be of very little interest when the order is higher than the fourth. I propose, after briefly noticing the results obtained by Kummer, Schlömilch, and Cauchy, to give the summations to which these integrals lead.

It will be convenient in what follows to write $e^{-a^2}=q$ (q being therefore less than unity), so that $a=\sqrt{\log\frac{1}{q}}$; a will, however, frequently be retained for the sake of brevity.

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The most obvious method of summing the series

$$1 + q^4 + q^9 + q^{16} + \ldots$$

is by means of the integral $\int_0^\infty e^{-x^2} \cos 2ax\,dx$ as follows:

$$q = e^{-a^2} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \cos 2ax\,dx;$$

therefore $q^{n^2} = e^{-n^2a^2} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \cos 2nax\,dx;$

therefore $1 + q + q^4 + q^9 + \ldots$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} (1 + \cos 2ax + \cos 4ax + \cos 6ax + \ldots)\,dx \ldots.(1).$$

The difficulty, however, at once presents itself that the sum of the series of cosines on the right-hand side must be indeterminate, as its usual analytical equivalent $\frac{1}{2}$ is here clearly inadmissible.

Kummer obtained the sum of the series from the integral, and was enabled to avoid the consideration of this difficulty by making use of the summation*

$$\cos 2\beta ax + v \cos 2(\beta - 1)ax + v^2 \cos 2(\beta - 2)ax + \ldots$$

$$= \frac{\cos 2\beta ax - v \cos 2(\beta + 1)ax}{1 - 2v \cos 2ax + v^2},$$

whence $q^{\beta^2} + vq^{(\beta-1)^2} + v^2q^{(\beta-2)^2} + \ldots$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \frac{\cos 2\beta ax - v \cos 2(\beta + 1)ax}{1 - 2v \cos 2ax + v^2}\,dx.$$

The general term of the series is $v^n q^{(\beta-n)^2} = v^n q^{\beta^2 + n - 2\beta n}$ which becomes $z^n q^{\beta^2 + n}$ on taking $z = vq^{-2\beta}$; thus we obtain the equation

$$1 + zq + z^2q^4 + z^3q^9 + \ldots$$

$$= \frac{2q^{-\beta^2}}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \frac{\cos 2\beta ax - zq^{2\beta} \cos 2(\beta + 1)ax}{1 - 2zq^{2\beta} \cos 2ax + z^2q^{4\beta}}\,dx$$

in which β is arbitrary, subject only to the condition that $v, = zq^{2\beta}$, must be less than unity. Since $\beta = 0$ would give

* *Crelle's Journal*, t. XVII., p. 220.

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the case (1), Kummer puts $\beta = \frac{1}{2}$, so that, taking $z = 1$,

$$1 + q + q^4 + q^9 + \ldots = \frac{2}{q^{\frac{1}{4}}\sqrt{\pi}} \int_0^\infty e^{-x^2} \frac{\cos ax - q\cos 3ax}{1 - 2q\cos 2ax + q^2}\, dx.$$

Similarly, by putting $z = -1$, the sum of the series $1 - q + q^4 - \ldots$ is obtained. The series $q + zq^9 + z^2q^{25} + \ldots$, $q^{\frac{1}{4}} + zq^{\frac{9}{4}} + z^2q^{\frac{25}{4}} + \ldots$ are also summed in the same manner.

Schlömilch's method* is equally ingenious, we have

$$re^{-\frac{a^2}{4}} + r^3e^{-\frac{9a^2}{4}} + r^5e^{-\frac{25a^2}{4}} + \ldots$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} (r\cos ax + r^3\cos 3ax + r^5\cos 5ax + \ldots)\, dx,$$

and the general term is $r^n e^{-\frac{(2n-1)^2}{4}a^2} = r^n e^{-n^2a^2 - \frac{1}{4}a^2 + na^2}$ which becomes $= q^{n^2+\frac{1}{4}}$, if we take $r = e^{-a^2} = q$.

The sum of the series of cosines can easily be shewn to be $\frac{(1-r)\, r\cos ax}{1 - 2r\cos 2ax + r^2}$, so that

$$q^{1+\frac{1}{4}} + q^{4+\frac{1}{4}} + q^{9+\frac{1}{4}} + \ldots = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \frac{(1-q)\, q\cos ax}{1 - 2q\cos 2ax + q^2}\, dx,$$

and therefore

$$q + q^4 + q^9 + \ldots = \frac{2}{\sqrt{\pi}} (1-q)\, q^{\frac{1}{4}} \int_0^\infty e^{-x^2} \frac{\cos ax\, dx}{1 - 2q\cos 2ax + q^2}.$$

Schlömilch also summed the series $q - q^4 + \ldots$ and $1 \pm \frac{1}{2}q^4 + \frac{1}{3}q^9 \pm \ldots$ in a similar manner.

Cauchy applied his theory of the 'principal value' of an integral directly to (1)†. His results, which include (1) as a particular case, may be stated as follows:

If $f(\alpha) = \int_0^\infty \phi(x)\cos \alpha x\, dx$, $F(\alpha) = \int_0^\infty \psi(x)\sin \alpha x\, dx$,

then $f(0) + zf(\alpha) + z^2f(2\alpha) + \ldots$

$$= \int_0^\infty \phi(x)\,(1 + z\cos ax + z^2\cos 2ax + \ldots)\, dx$$

$$= \int_0^\infty \phi(x) \frac{1 - z\cos ax}{1 - 2z\cos ax + z^2}\, dx \ldots\ldots (2).$$

* *Analytische Studien*, part I. p. 158.

† *Exercises de Mathématiques*, 1827, p. 148.

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$$zF(\alpha) + z^2F(2\alpha) + \ldots$$

$$= \int_0^\infty \psi(x)\,(z\sin\alpha x + z^2\sin 2\alpha x + \ldots)\,dx$$

$$= \int_0^\infty \psi(x)\,\frac{z\sin\alpha x}{1 - 2z\cos\alpha x + z^2}\,dx \ldots\ldots (3).$$

Now if we put $z = 1$, the integrals (2) and (3) become $\int_0^\infty \phi(x) \cdot \frac{1}{2} \cdot dx$ and $\frac{1}{2}\int_0^\infty \psi(x)\cot\frac{\alpha x}{2}\,dx$.

Cauchy, therefore, investigated the limiting values to which the integrals (2) and (3) tend as z approaches unity, and found that (2) has for its limit the quantity

$$\frac{\pi}{\alpha}\left\{\tfrac{1}{2}\phi(0) + \phi\left(\frac{2\pi}{\alpha}\right) + \phi\left(\frac{4\pi}{\alpha}\right) + \ldots\right\} - \tfrac{1}{2}f(0),$$

and that (3) becomes equal to $\frac{1}{2}\int_0^\infty \psi(x)\cot\frac{\alpha x}{2}\,dx$, on the understanding that when the quantity under the integral sign becomes infinite, the integral is supposed to have its principal value. Thus, by taking

$$f(\alpha) = \sqrt{\frac{\pi}{2}} \cdot e^{-\frac{\alpha^2}{4}} = \int_0^\infty e^{-\frac{x^2}{4}}\cos\alpha x\,dx,$$

in (2), we see that the equation to which (1) leads is

$$\tfrac{1}{2} + e^{-\pi^2} + e^{-4\pi^2} + e^{-9\pi^2} + \ldots = \frac{\sqrt{\pi}}{\alpha}\left\{\tfrac{1}{2} + e^{-\frac{\pi^2}{\alpha^2}} + e^{-\frac{4\pi^2}{\alpha^2}} + e^{-\frac{9\pi^2}{\alpha^2}} + \ldots\right\},$$

a theorem to which reference will be made further on.

From (3), since $\alpha e^{-\alpha^2} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}x\sin 2\alpha x\,dx$, we have

$$q + 2q^4 + 3q^9 + \ldots = \frac{1}{\alpha\sqrt{\pi}}\int_0^\infty xe^{-x^2}\cot\alpha x\,dx.$$

Cauchy's result, corresponding to that last written, is

$$q + q^4 + q^9 + \ldots = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\cot\alpha x\,dx,$$

but this appears inaccurate,* as it is not true that

$$e^{-\alpha^2} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\sin 2\alpha x\,dx.$$

* The integrals (7) Table 263, and (7) Table 406 of Prof. de Haan's *Nouvelles Tables d'Intégrales Définies* are thus rendered incorrect; as well as (22) Table 280 in the fourth volume of the *Amsterdam Transactions*.

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It may be remarked that the method of summation by a series of cosines is not so general as at first sight it may appear; as it is necessary for the correctness of the integral that the quantity subject to the integral sign should not be discontinuous, while discontinuity is the chief characteristic of a series of periodic functions. Thus, for instance, although

$$q - \tfrac{1}{4}q^4 + \tfrac{1}{9}q^9 - \ldots$$
$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \left(\cos 2ax - \frac{1}{2^2} \cos 4ax + \frac{1}{3^2} \cos 6ax + \ldots \right) dx,$$

still we do not from this equation obtain a summation of the left-hand series as a definite integral, for it is clearly impossible to replace $\cos 2ax - \frac{1}{2^2} \cos 4ax + \ldots$ by its sum $\frac{\pi^2}{12} - a^2x^2$, since this is only true between the limits $\frac{\pi}{2a}$ and $-\frac{\pi}{2a}$ of x.

The integrals $\int_0^\infty \frac{b \cos ax}{b^2 + x^2} dx$ and $\int_0^\infty \frac{x \sin ax}{b^2 + x^2} dx$ can be applied to the summation of geometrical series as follows:

$$e^{-ab} = \frac{2}{\pi} \int_0^\infty \frac{b \cos ax}{b^2 + x^2} dx \quad \ldots\ldots\ldots\ldots\ldots (4),$$

whence
$$q^{n^2} = e^{-n^2 a} = \frac{2}{\pi} \int_0^\infty \frac{n^2 \cos ax}{n^4 + x^2} dx \quad \ldots\ldots\ldots\ldots\ldots (5),$$

in which $q = e^{-a}$; that is, a and q are related by the equation

$$a = \log \frac{1}{q} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

The integral (4), as also the integral $\int_0^\infty \frac{x \sin ax}{b^2 + x^2} dx$, which is equal to the same quantity, viz. e^{-ab}, requires that ab should be positive; thus, as only positive values will be assigned to b, q (which $= e^{-a}$) must be less than unity. In describing Kummer's and Schlömilch's researches the relation between a and q was $a = \sqrt{\log \frac{1}{q}}$, so that $a^2 = \alpha$.

From (5)

$$q + q^4 + q^9 + \ldots = \frac{2}{\pi} \int_0^\infty \left(\frac{1^2}{1^4 + x^2} + \frac{2^2}{2^4 + x^2} + \frac{3^2}{3^4 + x^2} + \ldots \right) \cos ax \, dx$$
$$= \int_0^\infty \frac{e^{2\pi\sqrt{(2x)}} - 2e^{\pi\sqrt{(2x)}} \sin \pi\sqrt{(2x)} - 1}{e^{2\pi\sqrt{(2x)}} - 2e^{\pi\sqrt{(2x)}} \cos \pi\sqrt{(2x)} + 1} \frac{\cos ax}{\sqrt{(2x)}} dx \ldots\ldots (7).$$

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[The sum of the series under the integral sign, viz. $\Sigma_1^\infty \frac{n^2}{n^4+x^2}$ can be deduced from that of series $\Sigma_1^\infty \frac{1}{n^2+x}$; for we know that

$$\frac{1}{1^2+\beta^2}+\frac{1}{2^2+\beta^2}+\frac{1}{3^2+\beta^2}+\ldots=\frac{\pi}{2\beta}\,\frac{e^{\pi\beta}+e^{-\pi\beta}}{e^{\pi\beta}-e^{-\pi\beta}}-\frac{1}{2\beta^2}\,*\ldots(8),$$

and $$\frac{n^2}{n^4+\beta^4}=\frac{1}{2}\left(\frac{1}{n^2+\beta^2 i}+\frac{1}{n^2-\beta^2 i}\right)\{i=\sqrt{-1}\};$$

therefore

$$\Sigma_1^\infty \frac{n^2}{n^4+\beta^4}=\frac{1}{2}\left\{\frac{\pi}{2\beta\,\frac{1+i}{\sqrt{2}}}\cdot\frac{e^{\frac{\pi\beta+\pi\beta i}{\sqrt{2}}}+e^{-\frac{\pi\beta+\pi\beta i}{\sqrt{2}}}}{e^{\frac{\pi\beta+\pi\beta i}{\sqrt{2}}}-e^{-\frac{\pi\beta+\pi\beta i}{\sqrt{2}}}}-\frac{1}{2\beta^2 i}\right.$$

$$\left.+\frac{\pi}{2\beta\,\frac{1-i}{\sqrt{2}}}\cdot\frac{e^{\frac{\pi\beta-\pi\beta i}{\sqrt{2}}}+e^{-\frac{\pi\beta-\pi\beta i}{\sqrt{2}}}}{e^{\frac{\pi\beta-\pi\beta i}{\sqrt{2}}}-e^{-\frac{\pi\beta-\pi\beta i}{\sqrt{2}}}}+\frac{1}{2\beta^2 i}\right\}.$$

By expanding the imaginary factors by De Moivre's theorem, *i.e.* writing $e^{\frac{\pi\beta i}{\sqrt{2}}}=\cos\frac{\pi\beta}{\sqrt{2}}+i\sin\frac{\pi\beta}{2}$, we obtain, after some deductions,

$$\Sigma_1^\infty \frac{n^2}{n^4+\beta^4}=\frac{\pi}{2\beta\sqrt{2}}\,\frac{e^{\pi\beta\sqrt{2}}-e^{-\pi\beta\sqrt{2}}-2\sin(\pi\beta\sqrt{2})}{e^{\pi\beta\sqrt{2}}+e^{-\pi\beta\sqrt{2}}-2\cos(\pi\beta\sqrt{2})},$$

and putting $\beta=\sqrt{x}$ and substituting for the series in the integral, (7) is obtained].

Similarly, from the integral

$$q^{n^2}=\int_0^\infty \frac{x\sin ax}{n^4+x^2}\,dx\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

* This can be proved in several ways, but most simply thus:

$$\cot a\pi=\frac{1}{\pi a}+\frac{1}{\pi a-\pi}+\frac{1}{\pi a+\pi}+\frac{1}{\pi a-2\pi}+\frac{1}{\pi a+2\pi}+\ldots;$$

therefore $$\pi\cot a\pi=\frac{1}{a}+\frac{2a}{a^2-1}+\frac{2a}{a^2-2^2}+\ldots,$$

put $a=\beta i$ and

$$\frac{1}{\beta^2+1}+\frac{1}{\beta^2+2^2}+\ldots=-\frac{\pi}{2\beta i}\cot\beta\pi i-\frac{1}{2\beta^2}.$$

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we find $q+q^5+q^9+\ldots$

$$=\int_0^\infty \left\{\frac{1}{(\sqrt{2}x)^3}\cdot\frac{e^{\pi\sqrt{\pi x}}+2e^{\pi\sqrt{\pi x}}\sin\pi\sqrt{2x}-1}{e^{\pi\sqrt{\pi x}}-2e^{\pi\sqrt{\pi x}}\cos\pi\sqrt{2x}+1}-\frac{1}{\pi x^2}\right\}x\sin\alpha x\,dx,$$

but $\int_0^\infty \frac{\sin\alpha x}{x}\,dx=\frac{1}{2}\pi$, α being positive, so that

$$\tfrac{1}{2}+q+q^5+q^9+\ldots$$
$$=\int_0^\infty \frac{e^{\pi\sqrt{(\pi x)}}+2e^{\pi\sqrt{(\pi x)}}\sin\pi\sqrt{(2x)}-1}{e^{\pi\sqrt{(\pi x)}}-2e^{\pi\sqrt{(\pi x)}}\cos\pi\sqrt{(2x)}+1}\,\frac{\sin\alpha x}{\sqrt{(2x)}}\,dx;$$

for obtaining which result, the formula used is

$$\Sigma_1^\infty \frac{1}{n^4+\beta^4}=\frac{\pi}{2\beta^3\sqrt{2}}\,\frac{e^{\pi\beta\sqrt{2}}-e^{-\pi\beta\sqrt{2}}+2\sin(\pi\beta\sqrt{2})}{e^{\pi\beta\sqrt{2}}+e^{-\pi\beta\sqrt{2}}-2\cos(\pi\beta\sqrt{2})}-\frac{1}{2\beta^4}.$$

It will be noticed that in these expressions the hyperbolic sine and cosine enter symmetrically with the trigonometrical sine and cosine, so that it will be convenient to have a short notation to express the former. Let then, in accordance with Prof. Cayley's notation,

$$\sinh x=\tfrac{1}{2}(e^x-e^{-x}),$$
$$\cosh x=\tfrac{1}{2}(e^x+e^{-x}),$$

and the summations take the form

$$\Sigma_1^\infty \frac{n^2}{n^4+\beta^4}=\frac{\pi}{2\beta\sqrt{2}}\,\frac{\sinh(\pi\beta\sqrt{2})-\sin(\pi\beta\sqrt{2})}{\cosh(\pi\beta\sqrt{2})-\cos(\pi\beta\sqrt{2})}\quad\ldots(10),$$

$$\Sigma_1^\infty \frac{1}{n^4+\beta^4}=\frac{\pi}{2\beta^3\sqrt{2}}\,\frac{\sinh(\pi\beta\sqrt{2})+\sin(\pi\beta\sqrt{2})}{\cosh(\pi\beta\sqrt{2})+\cos(\pi\beta\sqrt{2})}-\frac{1}{2\beta^4}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

and the integrals become

$$q+q^5+q^9+\ldots$$
$$=\int_0^\infty \frac{\sinh\{\pi\sqrt{(2x)}\}-\sin\{\pi\sqrt{(2x)}\}}{\cosh\{\pi\sqrt{(2x)}\}-\cos\{\pi\sqrt{(2x)}\}}\,\frac{\cos\alpha x}{\sqrt{(2x)}}\,dx\ldots(12),$$

$$\tfrac{1}{2}+q+q^5+q^9+\ldots$$
$$=\int_0^\infty \frac{\sinh\{\pi\sqrt{(2x)}\}+\sin\{\pi\sqrt{(2x)}\}}{\cosh\{\pi\sqrt{(2x)}\}-\cos\{\pi\sqrt{(2x)}\}}\,\frac{\sin\alpha x}{\sqrt{(2x)}}\,dx\ldots(13).$$

It is evident, from the two examples which have just been ven, that we can always by means of either of the integrals

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(5) or (9) express $A_1q + A_2q^4 + A_3q^9 + \ldots$ as a definite integral whenever we can sum the series

$$\frac{A_1}{1^2+\beta^2} + \frac{A_2}{2^2+\beta^2} + \frac{A_3}{3^2+\beta^2} + \ldots,$$

for $$\frac{A_n n^2}{n^4+\beta^4} = \tfrac{1}{2}A_n \left\{\frac{1}{n^2-\beta^2 i} + \frac{1}{n^2+\beta^2 i}\right\},$$

and $$\frac{A_n}{n^4+\beta^4} = \frac{A_n}{2\beta^2 i}\left\{\frac{1}{n^2-\beta^2 i} - \frac{1}{n^2+\beta^2 i}\right\}.$$

The following are a few simple cases in which the series admit of summation:

$$\frac{1}{1^2+\beta^2} - \frac{1}{2^2+\beta^2} + \frac{1}{3^2+\beta^2} - \ldots = \frac{1}{2\beta^2} - \frac{\pi}{2\beta\sinh(\pi\beta)} \quad \ldots (14),$$

$$\frac{1}{1^2+\beta^2} - \frac{3}{3^2+\beta^2} + \frac{5}{5^2+\beta^2} - \ldots = \frac{\pi}{4\cosh(\frac{1}{2}\pi\beta)} \quad \ldots\ldots\ldots (15),$$

$$\frac{1}{1^2+\beta^2} - \frac{1}{3(3^2+\beta^2)} + \frac{1}{5(5^2+\beta^2)} - \ldots$$

$$= \frac{\pi}{4\beta^2}\left\{1 - \frac{1}{\cosh(\frac{1}{2}\pi\beta)}\right\} \quad \ldots\ldots\ldots\ldots (16),$$

$$\frac{1}{1^2(1^2+\beta^2)} + \frac{1}{2^2(2^2+\beta^2)} + \frac{1}{3^2(3^2+\beta^2)} + \ldots$$

$$= \frac{\pi^2}{6\beta^2} + \frac{1}{2\beta^4} - \frac{\pi}{2\beta^3}\frac{\cosh(\pi\beta)}{\sinh(\pi\beta)} \quad \ldots\ldots (17),$$

$$\frac{1}{1^2(1^2+\beta^2)} - \frac{1}{2^2(2^2+\beta^2)} + \frac{1}{3^2(3^2+\beta^2)} - \ldots$$

$$= \frac{\pi^2}{12\beta^2} - \frac{1}{2\beta^4} + \frac{\pi}{2\beta^3}\frac{1}{\sinh(\pi\beta)} \quad \ldots\ldots (18),$$

these may be deduced from the series

$$\frac{\cos x}{1^2+\beta^2} + \frac{\cos 2x}{2^2+\beta^2} + \frac{\cos 3x}{3^2+\beta^2} + \ldots = \frac{\pi}{2\beta}\frac{\cosh\{\beta(\pi-x)\}}{\sinh(\pi\beta)} - \frac{1}{2\beta^2},$$

(which is obtained by Fourier's theorem) by differentiation and integration, but (17) and (18) may be obtained more simply by subtracting (8) and (14) from $\frac{1}{1^2} + \frac{1}{2^2} + \ldots (= \frac{1}{6}\pi^2)$ and $\frac{1}{1^2} - \frac{1}{2^2} + \ldots (= \frac{1}{12}\pi^2)$ respectively.

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The integral-summations to which (14)...(18) lead, are

$$q - q^4 + q^9 - \ldots$$
$$= \int_0^\infty \frac{\cosh(\pi\sqrt{\frac{1}{2}x})\sin(\pi\sqrt{\frac{1}{2}x}) - \sinh(\pi\sqrt{\frac{1}{2}x})\cos(\pi\sqrt{\frac{1}{2}x})}{\cosh\{\pi\sqrt{(2x)}\} - \cos\{\pi\sqrt{(2x)}\}} \frac{\cos ax}{\sqrt{(2x)}}\,dx$$
..... (19),

$$\tfrac{1}{2} - q + q^4 - q^9 + \ldots$$
$$= \int_0^\infty \frac{\cosh(\pi\sqrt{\frac{1}{2}x})\sin(\pi\sqrt{\frac{1}{2}x}) + \sinh(\pi\sqrt{\frac{1}{2}x})\cos(\pi\sqrt{\frac{1}{2}x})}{\cosh\{\pi\sqrt{(2x)}\} - \cos\{\pi\sqrt{(2x)}\}} \frac{\sin ax}{\sqrt{(2x)}}\,dx$$
..... (20),

$$q - 3q^9 + 5q^{25} - \ldots$$
$$= \frac{1}{2}\int_0^\infty \frac{\cosh(\frac{1}{2}\pi\sqrt{\frac{1}{2}x})\cos(\frac{1}{2}\pi\sqrt{\frac{1}{2}x})}{\cosh(\pi\sqrt{\frac{1}{2}x}) + \cos(\pi\sqrt{\frac{1}{2}x})} \cos ax\,dx \ldots\ldots\ldots (21),$$

$$q - 3q^9 + 5q^{25} - \ldots$$
$$= \frac{1}{2}\int_0^\infty \frac{\sinh(\frac{1}{2}\pi\sqrt{\frac{1}{2}x})\sin(\frac{1}{2}\pi\sqrt{\frac{1}{2}x})}{\cosh(\pi\sqrt{\frac{1}{2}x}) + \cos(\pi\sqrt{\frac{1}{2}x})} \sin ax\,dx \ldots\ldots\ldots (22),$$

$$q - \tfrac{1}{3}q^9 + \tfrac{1}{5}q^{25} - \ldots$$
$$= \frac{1}{2}\int_0^\infty \frac{\sinh(\frac{1}{2}\pi\sqrt{\frac{1}{2}x})\sin(\frac{1}{2}\pi\sqrt{\frac{1}{2}x})}{\cosh(\pi\sqrt{\frac{1}{2}x}) + \cos(\pi\sqrt{\frac{1}{2}x})} \frac{\cos ax}{x}\,dx \ldots\ldots\ldots (23),$$

$$\tfrac{1}{4}\pi - q + \tfrac{1}{3}q^9 - \tfrac{1}{5}q^{25} + \ldots$$
$$= \frac{1}{2}\int_0^\infty \frac{\cosh(\frac{1}{2}\pi\sqrt{\frac{1}{2}x})\cos(\frac{1}{2}\pi\sqrt{\frac{1}{2}x})}{\cosh(\pi\sqrt{\frac{1}{2}x}) + \cos(\pi\sqrt{\frac{1}{2}x})} \frac{\sin ax}{x}\,dx \ldots\ldots\ldots (24),$$

$$q + \tfrac{1}{4}q^4 + \tfrac{1}{9}q^9 + \ldots$$
$$= \int_0^\infty \left[\frac{\sinh\{\pi\sqrt{(2x)}\} + \sin\{\pi\sqrt{(2x)}\}}{\cosh\{\pi\sqrt{(2x)}\} - \cos\{\pi\sqrt{(2x)}\}} \frac{1}{x\sqrt{(2x)}} - \frac{1}{\pi x^2}\right]\cos ax\,dx$$
..... (25),

$$-\tfrac{1}{6}\pi^2 + q + \tfrac{1}{4}q^4 + \tfrac{1}{9}q^9 + \ldots$$
$$= \int_0^\infty \frac{\sin\{\pi\sqrt{(2x)}\} - \sinh\{\pi\sqrt{(2x)}\}}{\cosh\{\pi\sqrt{(2x)}\} - \cos\{\pi\sqrt{(2x)}\}} \frac{\sin ax}{x\sqrt{(2x)}}\,dx \ldots\ldots (26),$$

$$q - \tfrac{1}{4}q^4 + \tfrac{1}{9}q^9 - \ldots$$
$$= \int_0^\infty \left[\frac{\cosh(\pi\sqrt{\frac{1}{2}x})\sin(\pi\sqrt{\frac{1}{2}x}) + \sinh(\pi\sqrt{\frac{1}{2}x})\cos(\pi\sqrt{\frac{1}{2}x})}{\cos\{\pi\sqrt{(2x)}\} - \cosh\{\pi\sqrt{(2x)}\}} \frac{1}{x\sqrt{(2x)}} + \frac{1}{\pi x^2}\right]\cos ax\,dx \ldots\ldots (27),$$

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$$-\tfrac{1}{12}\pi^2+q-\tfrac{1}{4}q^4+\tfrac{1}{9}q^9-\ldots$$

$$=\int_0^\infty \frac{\sinh(\pi\sqrt{\tfrac{1}{2}x})\cos(\pi\sqrt{\tfrac{1}{2}x})-\cosh(\pi\sqrt{\tfrac{1}{2}x})\sin(\pi\sqrt{\tfrac{1}{2}x})}{\cosh\{\pi\sqrt{(2x)}\}-\cos\{\pi\sqrt{(2x)}\}}$$

$$\times\frac{\sin\alpha x}{x\sqrt{(2x)}}\,dx\ldots\ldots\ldots\ldots(28).$$

Of these, (19) and (20) follow from (14), (21) and (22) from (15), (23) and (24) from (16), (25) and (26) from (17), (27) and (28) from (18), by the aid of the integrals (5) and (9).

It is unnecessary to give any details of the work by which these integrals are deduced, as it resembles in all the cases that indicated above for the series $q+q^4+q^9+\ldots$. The reductions were tedious and troublesome. The constant which occurs several times on the left-hand side enters into the equations through replacing $\int_0^\infty \frac{\sin\alpha x}{x}\,dx$ by $\frac{\pi}{2}$.

The series-summations (which are omitted for brevity) can easily be deduced from the integrals if required; thus, for example from (24),

$$q-\tfrac{1}{3}q^9+\ldots$$

$$=\int_0^\infty\left\{-\frac{\cosh(\frac{1}{2}\pi\sqrt{\tfrac{1}{2}x})\cos(\frac{1}{2}\pi\sqrt{\tfrac{1}{2}x})}{\cosh(\pi\sqrt{\tfrac{1}{2}x})+\cos(\pi\sqrt{\tfrac{1}{2}x})}\cdot\frac{1}{2x^2}+\frac{1}{2x^2}\right\}x\sin\alpha x\,dx,$$

we see that

$$\frac{1}{1^4+\beta^4}-\frac{1}{3}\cdot\frac{1}{3^4+\beta^4}+\ldots=\frac{\pi}{4}\left\{\frac{1}{\beta^4}-\frac{1}{\beta^4}\,\frac{\cosh\left(\frac{\pi\beta}{2\sqrt{2}}\right)\cos\left(\frac{\pi\beta}{2\sqrt{2}}\right)}{\cosh\left(\frac{\pi\beta}{\sqrt{2}}\right)+\cos\left(\frac{\pi\beta}{\sqrt{2}}\right)}\right\}.$$

For the summation of geometrical series of the third order (such as $q+q^8+q^{27}+\ldots$), the integral (9) is alone appropriate; for (5) gives

$$\int_0^\infty \frac{n^3\cos\alpha x}{n^6+x^3}\,dx=\frac{\pi}{2}\,q^{n^3},$$

and $\frac{n^3}{n^6+\beta^6}$ cannot be resolved into partial fractions of the form $\frac{A}{n^3+B}$; from (9), however, we have

$$\int_0^\infty \frac{x\sin\alpha x}{n^6+x^3}\,dx=\frac{\pi}{2}\,q^{n^3},$$

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and $\frac{1}{n^6+\beta^6}$ can be resolved into partial fractions of the required form, viz.

$$\frac{1}{n^6+\beta^6} = \frac{1}{3\beta^4}\left\{\frac{1}{n^2+\beta^2} - \frac{1}{2}\frac{1+i\sqrt{3}}{n^2-\beta^2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{2}\frac{1-i\sqrt{3}}{n^2-\beta^2\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}\right\}.$$

Now $$-\beta^2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) = \left(\beta\frac{i\sqrt{3}-1}{2}\right)^2,$$

whence from (8)

$$\Sigma_1^\infty \frac{1+i\sqrt{3}}{n^2-\beta^2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}$$

$$= \frac{\pi}{2\beta}\frac{i\{\sqrt{3}\sinh(\pi\beta)-\sin(\pi\beta\sqrt{3})\}-\sinh(\pi\beta)-\sqrt{3}\sin(\pi\beta\sqrt{3})}{\cosh(\pi\beta)-\cos(\pi\beta\sqrt{3})} + \frac{1}{\beta^2},$$

therefore

$$\Sigma_1^\infty \frac{1}{2}\left\{\frac{1+i\sqrt{3}}{n^2-\beta^2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} + \frac{1-i\sqrt{3}}{n^2-\beta^2\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}\right\}$$

$$= -\frac{\pi}{2\beta}\frac{\sinh(\pi\beta)+\sqrt{3}\sin(\pi\beta\sqrt{3})}{\cosh(\pi\beta)-\cos(\pi\beta\sqrt{3})} + \frac{1}{\beta^2},$$

and $\Sigma_1^\infty \frac{3\beta^4}{n^6+\beta^6}$

$$= \frac{\pi}{2\beta}\left\{\frac{\cosh(\pi\beta)}{\sinh(\pi\beta)} + \frac{\sinh(\pi\beta)+\sqrt{3}\sin(\pi\beta\sqrt{3})}{\cosh(\pi\beta)-\cos(\pi\beta\sqrt{3})}\right\} - \frac{3}{2\beta^2},$$

whence $\Sigma_1^\infty \frac{1}{n^6+\beta^6}$

$$= \frac{\pi}{6\beta^5}\frac{\sqrt{3}\sinh(\pi\beta)\sin(\pi\beta\sqrt{3})-\cosh(\pi\beta)\cos(\pi\beta\sqrt{3})+\cosh(2\pi\beta)}{\sinh(\pi\beta)\{\cosh(\pi\beta)-\cos(\pi\beta\sqrt{3})\}} - \frac{1}{2\beta^6},$$

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$$\text{and } q+q^8+q^{27}+\ldots=\frac{1}{2}-\frac{1}{3}\int_0^\infty\left\{\frac{1}{\sinh(\pi x^{\frac{1}{2}})}\right.$$

$$\left.+\frac{\sinh(\frac{1}{2}\pi x^{\frac{1}{2}})\cos(\frac{1}{2}\pi x^{\frac{1}{2}}\sqrt{3})+\sqrt{3}\cosh(\frac{1}{2}\pi x^{\frac{1}{2}})\sin(\frac{1}{2}\pi x^{\frac{1}{2}}\sqrt{3})}{\cosh(\pi x^{\frac{1}{2}})-\cos(\pi x^{\frac{1}{2}}\sqrt{3})}\right\}$$

$$\times\frac{\sin\alpha x}{x^{\frac{3}{2}}}\,dx\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29).$$

In a similar manner by the aid of (14) we can prove that

$$q-q^8+q^{27}-\ldots=\frac{1}{2}-\frac{1}{3}\int_0^\infty\left\{\frac{1}{\sinh(\pi x^{\frac{1}{2}})}\right.$$

$$\left.+\frac{\sinh(\frac{1}{2}\pi x^{\frac{1}{2}})\cos(\frac{1}{2}\pi x^{\frac{1}{2}}\sqrt{3})+\sqrt{3}\cosh(\frac{1}{2}\pi x^{\frac{1}{2}})\sin(\frac{1}{2}\pi x^{\frac{1}{2}}\sqrt{3})}{\cosh(\pi x^{\frac{1}{2}})-\cos(\pi x^{\frac{1}{2}}\sqrt{3})}\right\}$$

$$\times\frac{\sin\alpha x}{x^{\frac{3}{2}}}\,dx\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30).$$

From the formulæ (15) ... (18) we could, of course, obtain the sums of the series $q-3q^{27}+5q^{125}-\ldots$, $q-\frac{1}{3}q^{27}+\ldots$, and $q\pm\frac{1}{4}q^8+\ldots$.

Both of the integrals (5) and (9) are available for the summation of geometrical series of the fourth order; as indeed they are for any even order, (9) alone being available for odd orders.

Since $\frac{1}{n^{2r}+\beta^{2r}}$ can always be resolved into r partial fractions of the form $\frac{A}{n^2+B}$, it follows, that we can always by means of (5) effect an integral summation of the series $A_1q^{1^r}+A_2q^{2^r}+A_3q^{3^r}+\ldots$, whenever

$$\frac{A_1}{1^2+\beta^2}+\frac{A_2}{2^2+\beta^2}+\frac{A_3}{3^2+\beta^2}+\ldots$$

can be summed, and since $\frac{n^r}{n^{2r}+\beta^{2r}}$ can be resolved into partial fractions of the required form when r is even, (9) also gives an integral in this case.

The results, however, (which I have obtained as far as the fourth order inclusive) become, as before remarked, so complicated, and their calculation and reduction is so tedious, that it does not seem worth while to work out any of the integrals for higher orders.

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By the aid of the integral*

$$\int_0^\infty e^{a\cos bx}\cos(a\sin bx)\,\frac{c\,dx}{c^2+x^2} = \frac{\pi}{2}e^{ae^{-bc}} \quad \ldots\ldots (31),$$

we may derive the summations of an interesting class of series. Thus, putting $e^a = q$ and $e^{-b} = p$, $\left(\text{so that } a = \log q,\ b = \log\frac{1}{p}\right)$, but retaining a and b under the integral sign for brevity,

$$\frac{2}{\pi}\int_0^\infty e^{a\cos bx}\cos(a\sin bx)\,\frac{c\,dx}{c^2+x^2} = q^{pc} \quad \ldots\ldots (32).$$

In (31) a may be any quantity, but b and c must be such, that bc is positive, therefore q may be what we please, but p must be less than unity.

Thus, for example, we have from (14)

$$q^p - \tfrac{1}{2}q^{p^2} + \tfrac{1}{3}q^{p^3} - \ldots = \int_0^\infty \left\{\frac{1}{\pi x^2} - \frac{1}{x\sinh(\pi x)}\right\} e^{a\cos bx}\cos(a\sin bx)\,dx,$$

from (17) $q^p + \frac{1}{2^3}q^{p^2} + \frac{1}{3^3}q^{p^3} + \ldots$

$$= \int_0^\infty \left\{\frac{1}{3x^2} + \frac{1}{\pi x^4} - \frac{1}{x^3}\,\frac{\cosh(\pi x)}{\sinh(\pi x)}\right\} e^{a\cos bx}\cos(a\sin bx)\,dx,$$

from (10) $q^p + \frac{1}{4}q^{p^4} + \frac{1}{9}q^{p^9} + \ldots$

$$= \int_0^\infty \frac{\sinh\{\pi\sqrt{(2x)}\} - \sin\{\pi\sqrt{(2x)}\}}{\cosh\{\pi\sqrt{(2x)}\} - \cos\{\pi\sqrt{(2x)}\}}\, e^{a\cos bx}\cos(a\sin bx)\,\frac{dx}{\sqrt{(2x)}}.$$

It is unnecessary to give any more applications of the integral (32) to summations, as from what has been said, it is clear that it enables us to express as a definite integral the sum of any series whose n^{th} term is

$$A_n q^{n^r},$$

(q being any quantity, p any quantity numerically less than unity, and n and r integers), if we know the summation of

$$\Sigma_1^\infty \frac{A_n}{n^2+\beta^2}.$$

* This integral can be deduced from $\int_0^\infty \frac{c\cos bx}{c^2+x^2}\,dx = \frac{\pi}{2}e^{-bc}$; for

$$\int_0^\infty \left(1 + a\cos bx + \frac{a^2\cos 2bx}{1.2} + \ldots\right)\frac{c\,dx}{c^2+x^2} = \frac{\pi}{2}\left\{1 + ae^{-bc} + \frac{a^2}{1.2}e^{-2bc} + \ldots\right\},$$

but $1 + a\cos bx + \frac{a^2\cos 2bx}{1.2} + \ldots = e^{a\cos bx}\cos(a\sin bx)$, and this is true without any discontinuity for all values of x, whence the result follows.

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When c is infinite, (since $p<1$) $q^{\frac{x}{c}}$ has unity for its limit, so that for the series to be convergent, the coefficients A_n, A_{n+1} must converge.

The integrals

$$\int_0^\infty e^{a\cos bx}\sin(a\sin bx+gx)\frac{xdx}{c^2+x^2}=\frac{\pi}{2}e^{-gc-ae^{-bc}},$$

$$\int_0^\infty e^{a\cos bx}\cos(a\sin bx+gx)\frac{dx}{c^2+x^2}=\frac{\pi}{2c}e^{-gc+ae^{-bc}},$$

are also suitable for the summation of the same class of series.

From the integral

$$\int_0^\infty e^{-x^2}\cos 2bx\,dx=\frac{\sqrt{\pi}}{2}e^{-b^2}\ldots\ldots\ldots\ldots(33),$$

we can deduce in a manner similar to that in which (13) was obtained, that

$$\int_0^\infty e^{-x^2+a\cos 2bx}\cos(a\sin 2bx)\,dx=\frac{\sqrt{\pi}}{2}e^{ae^{-b^2}},$$

which is not of use for the summation of series.

The integral $\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,dx=\frac{\sqrt{\pi}}{2}e^{-2a}$ affords a means of summing a new class of series.

For putting $4a^2=a$ we have

$$\int_0^\infty e^{-x^2}e^{-\frac{a}{4x^2}}\,dx=\frac{\sqrt{\pi}}{2}e^{-\sqrt{a}},$$

whence

$$\int_0^\infty e^{-x^2}f(e^{-\frac{a}{4x^2}})\,dx=\frac{\sqrt{\pi}}{2}f(e^{-\sqrt{a}})\ldots\ldots\ldots(34),$$

$f(x)$ being any function expansible in ascending powers of x. The integral (34) is worthy of notice, as there are but few definite integrals, admitting an arbitrary function under the integral sign, that can be evaluated.

From (34) we deduce at once

$$1+q^{\sqrt{1}}+q^{\sqrt{2}}+q^{\sqrt{3}}+\ldots=\frac{2}{\sqrt{\pi}}\int_0^\infty\frac{e^{-x^2}dx}{1-e^{-\frac{a}{4x^2}}},$$

$$1-q^{\sqrt{1}}+q^{\sqrt{2}}-q^{\sqrt{3}}+\ldots=\frac{2}{\sqrt{\pi}}\int_0^\infty\frac{e^{-x^2}dx}{1+e^{-\frac{a}{4x^2}}}$$

$$\pm q^{\sqrt{1}}-\tfrac{1}{2}q^{\sqrt{2}}\pm\tfrac{1}{3}q^{\sqrt{3}}-\ldots=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\log(1\pm e^{-\frac{a}{4x^2}})\,dx,$$

&c. = &c.

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q being equal to $e^{-\sqrt{x}}$; so that if we can sum $\Sigma_1^\infty A_n q^n$, we can express $\Sigma_1^\infty A_n q^{\sqrt{n}}$ as a definite integral, the only case requiring care being when the sum of $\Sigma_1^\infty A_n q^n$ is ambiguous as, *e.g.*, when

$$\tan^{-1} e^{-\frac{a}{\omega^2}} = e^{-\frac{a}{\omega^2}} - \tfrac{1}{3} e^{-\frac{3a}{\omega^2}} + \tfrac{1}{5} e^{-\frac{5a}{\omega^2}} - \dots,$$

in which the arc intermediate to 0 and $\frac{1}{2}\pi$ must be taken. The integral may also be written by making an evident substitution in the form

$$\int_0^\infty e^{-\frac{ax^2}{4} - \frac{1}{x^2}} dx = \sqrt{\frac{\pi}{a}} . e^{-\sqrt{a}},$$

so that we can deduce $\Sigma_1^\infty A_n \frac{q^{\sqrt{n}}}{\sqrt{n}}$ from $\Sigma_1^\infty A_n q^n$; and writing (33) in the form

$$\int_0^\infty e^{-ax^2} \cos 2bx\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{a}} \quad \dots\dots\dots(35),$$

we see that $\Sigma_1^\infty A_n \frac{q^{\frac{1}{n}}}{\sqrt{n}}$ can be expressed as a definite integral if $\Sigma_1^\infty A_n q^n$ can be summed; thus, for example,

$$1 - \frac{1}{\sqrt{2}} q^{\frac{1}{2}} + \frac{1}{\sqrt{3}} q^{\frac{1}{3}} - \dots = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\cos 2bx\, dx}{1 + e^{-x^2}},$$

where $q = e^{-b^2}$ so that $b = \sqrt{\log \frac{1}{q}}$.

The integral (35) leads us back to the consideration of the series which Cauchy obtained from (1), viz.

$$\tfrac{1}{2} + e^{-a^2} + e^{-4a^2} + e^{-9a^2} + \dots = \frac{\sqrt{\pi}}{a} \left\{ \tfrac{1}{2} + e^{-\frac{\pi^2}{a^2}} + e^{-\frac{4\pi^2}{a^2}} + e^{-\frac{9\pi^2}{a^2}} + \dots \right\}$$
$$\dots\dots\dots\dots\dots\dots(36).$$

This theorem Cauchy speaks of as having appeared to Laplace of so much interest, that the latter verified it numerically when a was small. Cauchy also remarks that Poisson had obtained it also by considering the decomposition of a definite integral into singular integrals. Another investigation of it is to be found in Schlömilch's *Analytische Studien*, part II., p. 30.

Cauchy proved a more general theorem, viz.

$$e^{-x^2} + e^{-(x-a)^2} + e^{-(x+a)^2} + e^{-(x-2a)^2} + e^{-(x+2a)^2} + \dots$$
$$= \frac{2\sqrt{\pi}}{a} \left\{ \tfrac{1}{2} + e^{-\frac{\pi^2}{a^2}} \cos\frac{2\pi x}{a} + e^{-\frac{4\pi^2}{a^2}} \cos\frac{4\pi x}{a} + \dots \right\} \dots\dots(37).$$

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This result is also obtained by Sir W. Thomson in a far more natural manner, viz. from Fourier's theorem,* and another proof depending on elliptic functions, by Professor Cayley, is added.

The integral $\int_0^\infty xe^{-a^2} \cot ax\,dx$, shown in this paper to be equal to $a\sqrt{\pi}\,\Sigma_1^\infty ne^{-n^2a^2}$, has, I have since found, been evaluated by Prof. de Haan (*Nouvelles Tables* (16), Table 362), who finds that it is equal to $-a\sqrt{\pi}\,\Sigma_1^\infty ne^{-n^2a^2}$, a result identical in form, but of different sign from that given above. From the reference it is clear that Prof. de Haan obtained his evaluation by differentiating the integral

$$\int_0^\infty e^{-x^2} \log(\sin^2 ax)\,dx = \sqrt{\pi}\left\{-\log 2 + \Sigma_1^\infty \frac{1}{n} e^{-n^2a^2}\right\}$$

with respect to a.

This last integral was obtained by Schlömilch (*Analytische Studien*, part I., p. 159), in the form

$$\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-a^2} \log(4\sin^2 ax)\,dx = e^{-a^2} + \tfrac{1}{2}e^{-4a^2} + \tfrac{1}{3}e^{-9a^2} + \ldots .$$

The error of sign is caused by Schlömilch having taken the sum of the series

$$r\cos z + \tfrac{1}{2}r^2\cos 2z + \ldots \text{ to be } \tfrac{1}{2}\log(1 - 2r\cos z + r^2),$$

instead of $-\frac{1}{2}\log(1 - 2r\cos z + r^2)$. The integral in the formula marked (6) in Schlömilch's work has also a wrong sign for the same reason. In the Tables, the two errors in (7) and (8), Table 467, are corrected in the errata; but the integrals (15) and (16), Table 362, which were deduced from them, have not been corrected. The values of these last two integrals should have their signs changed. The value of the integral (10), Table 467, has also a wrong sign.

In the first edition of the Tables (*Amsterdam Transactions*, vol. IV.), the corresponding integrals are (12), (13), (14), Table 439, and (20), (21), Table 388, all of which are wrong in sign; in the last two the reference also is inaccurate.

* *Quarterly Journal*, vol. I., p. 316.

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ON A THEOREM RELATING TO EIGHT POINTS ON A CONIC.

By Professor CAYLEY.

THE following is a known theorem:

"In any octagon inscribed in a conic, the two sets of alternate sides intersect in the 8 points of the octagon and in 8 other points lying in a conic."

In fact the two sets of sides are each of them a quartic curve, hence any quartic curve through 13 of the 8 + 8 points passes through the remaining 3 points: but the original conic together with a conic through 5 of the 8 new points form together such a quartic curve; and hence the remaining 3 of the new points (inasmuch as obviously they are not situate on the original conic) must be situate on the conic through the 5 new points, that is the 8 new points must lie on a conic.

We may without loss of generality take $(\alpha_1^2, \alpha_1, 1)$, $(\alpha_2^2, \alpha_2, 1)$, ...$(\alpha_8^2, \alpha_8, 1)$, as the coordinates (x, y, z) of the 8 points of the octagon; and obtain hereby an *à posteriori* verification of the theorem, by finding the equation of the conic through the 8 new points: the result contains cylical expressions of an interesting form.

Calling the points of the octagon 1, 2, 3, 4, 5, 6, 7, 8 the 8 new points are

12.45, 23.56, 34.67, 45.78, 56.81, 67.12, 78.23, 81.34,

viz. 12.45 is the intersection of the lines 12 and 45; and so on. The 8 points lie on a conic, the equation of which is to be found.

The equation of the line 12 is

$$x - (\alpha_1 + \alpha_2)\, y + \alpha_1\alpha_2 z = 0,$$

or as it is convenient to write it

$$x - (1 + 2)\, y + 12 . z,$$

viz. 1, 2, &c., are for shortness written in place of α_1, α_2, &c. respectively.

The coordinates of the point 12.45 are consequent proportional to the terms of

$$1, \quad -(1+2), \quad 12,$$
$$1, \quad -(4+5), \quad 45,$$

or say they are as

$$12\,(4+5) - 45\,(1+2) : 12 - 45 : 1 + 2 - (4+5).$$

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The equation of the line (12.45)(23.56) which joins the points 12.45 and 23.56 thus is

$$\begin{vmatrix} x, & y, & z \\ 12(4+5)-45(1+2), & 12+45, & (1+2)-(4+5) \\ 23(5+6)-56(2+3), & 23-56, & (2+3)-(5+6) \end{vmatrix} = 0,$$

where the determinant vanishes identically if $2-5=0$ ($\alpha_2-\alpha_5=0$); it in fact thereby becomes

$$\begin{vmatrix} x, & y, & z \\ 2^2(1-4), & 2(1-4), & (1-4) \\ 2^2(3-6), & 3(3-6), & (3-6) \end{vmatrix},$$

which is $=0$; the determinant divides therefore by $2-5$; the coefficient of x is easily found to be

$$=(2-5)(12-23+34-45+56-61),$$

and so for the other terms; and omitting the factor $2-5$ the equation is

$$x\{12-23+34-45+56-61\}$$
$$-y\{12(4+5)-23(5+6)+34(6+1)-45(1+2)+56(2+3)-61(3+4)\}$$
$$+z\{1234-2345+3456-4561+5612-6123\}=0.$$

There is now not much difficulty in forming the equation of the required conic; viz. this is

$$(2-8)\{x-(6+7)y+67z\}$$
$$\left\{\begin{array}{l} x[12-23+34-45+56-61] \\ -y[12(4+5)-23(5+6)+34(6+1)-45(1+2)+56(2+3)-61(3+4)] \\ +z[1234-2345+3456-4561+5612-6123] \end{array}\right\}$$
$$+(6-8)\{x-(1+2)y+12z\}$$
$$\left\{\begin{array}{l} x[23-34+45-56+67-72] \\ -y[23(5+6)-34(6+7)+45(7+2)-56(2+3)+67(3+4)-72(4+5)] \\ +z[2345-3456+4567-5672+6723-7234] \end{array}\right\}$$
$$=0.$$

In fact this equation written with an indeterminate coefficient λ, say for shortness, thus

$$67[(12.45)(23.56)]=\lambda 12.[(23.56)(34.67)]=0,$$

is the general equation of the conic through the 4 points 12.67, 34.67, 12.45, and 23.56; and by making the conic

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pass through 1 of the remaining 4 of the 8 points, I succeeded in finding the value $\lambda = \frac{6-8}{2-8}$, so that the conic in question passes through 5 of the 8 points, and is therefore by the theorem the conic through the 8 points. But as thus written down the equation contains the extraneous factor $2-6$, as appears at once by the observation that the left-hand side on writing therein $6 = 2\ (\alpha_6 = \alpha_2)$ becomes identically $= 0$; the value in fact is

$$-(2-8)[x-(2+7)y+27z](23-34+45-52)[x-(1+2)y+12z]$$
$$+(2-8)[x-(1+2)y+12z](23-34+45-52)[x-(2+7)y+27z]$$

which is $= 0$; there is consequently the factor $2-6$ to be rejected, and throwing this out the equation assumes a symmetrical form in regard to the 8 symbols $1, 2, 3, 4, 5, 6, 7, 8$. The coefficient of x^4 is very easily found to be

$$= (2-6)\ (12-23+34-45+56-67+78-81),$$

and similarly that of z^4 to be

$$= (2-6)\ \{123456 - 234567 + 345678 - 456781 + 567812$$
$$- 678123 + 781234 - 812345\},$$

those of the other terms are somewhat more difficult to calculate; but the final result, throwing out the factor $(2-6)$, and introducing an abbreviated notation

$$\Sigma 12 = (12-23+34-45+56-67+78-81),$$

and the like in other cases, is found to be

$$\begin{aligned} & x^4 \,.\ \Sigma 12 \\ & + y^4 \,.\ [\Sigma 12\ (4+5)\ (6+7) - \tfrac{1}{2}\Sigma 1256] \\ & + z^4 \,.\ \Sigma 123456 \\ & - yz \,.\ \Sigma 16\ (234+235+245+345) \\ & + zx\ [\Sigma 1234 \qquad\qquad + \tfrac{1}{2}\Sigma 1256] \\ & - xy \,.\ \Sigma 12\ (4+5+6+7) \qquad\qquad = 0, \end{aligned}$$

where it is to be observed that $\Sigma 1256$ consists of 4 distinct terms each twice repeated: $\frac{1}{2}\Sigma 1256$ consists therefore of these 4 terms; and in the coefficient of y^4 they destroy 4 of the 32 terms of $\Sigma 12\ (4+5)\ (6+7)$ so that the coefficient of y^4 contains $32-4, = 28$ terms. In the coefficient of zx there is no destruction, and this contains therefore $12+4, = 16$ terms.

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ON THE ANALOGUES IN THE THEORY OF QUADRICS TO SOME KNOWN PROPERTIES IN THE THEORY OF CONICS.

By R. Townsend, M.A., F.R.S.

1°. When, in geometry of two dimensions, two conics S_1 and S_2 have a common centre and axes, it is evident, from the symmetry in the four quadrants determined by the axes, that the eight tangents at their four common points touch a common conic E, and that the eight points of contact of their four common tangents lie on a common conic F; the two associated conics E and F having the same common centre and axes with the two original conics S_1 and S_2.

Hence at once, by homographic transformation, for any two conics S_1 and S_2 having a common self-reciprocal triangle (that is for any two conics whatever), the eight tangents at their four common points touch a common conic E, and the eight points of contact of their four common tangents lie on a common conic F; the two associated conics E and F having the same common self-reciprocal triangle with the two original conics S_1 and S_2.*

2°. Analogously, in geometry of three dimensions, when three quadrics S_1, S_2, S_3 have a common centre and axes, it is evident, from the symmetry in the eight octants determined by the axes, that the twenty-four tangent planes at their eight common points touch a common quadric E, and that the twenty-four points of contact of their eight common tangent planes lie on a common quadric F; the two associated quadrics E and F having the same common centre and axes with the three original quadrics S_1, S_2, S_3.

Hence at once, by homographic transformation, for any three quadrics S_1, S_2, S_3 having a common self-reciprocal tetrahedron, the twenty-four tangent planes at their eight common points touch a common quadric E, and the twenty-four points of contact of their eight common tangent planes lie on a common quadric F; the two associated quadrics E

* See Salmon's *Conic Sections*, Edt. 5, Arts. 334 and 335. Where these properties (due originally to himself) are proved directly, and the equations of the two remarkable conics E and F (to the importance of which in the theory of two conics he was the first to direct attention), obtained in a very simple and instructive manner.

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and F having the same common self-reciprocal tetrahedron with the three original quadrics S_1, S_2, S_3.

3°. It is again evident, from the same aforesaid symmetry in the eight octants determined by the three axes, that, for three quadrics S_1, S_2, S_3 having a common centre and axes, the twenty-four tangent lines to two of them at their eight common points touch a common quadric H, and the twenty-four tangent lines to two of them in their eight common tangent planes touch a common quadric K; the two associated quadrics H and K having, like the two E and F, the same common centre and axes with the original three S_1, S_2, S_3.

Hence again, by homographic transformation, for any three quadrics S_1, S_2, S_3 having a common self-reciprocal tetrahedron, the twenty-four double tangents at their eight common points touch a common quadric H, and the twenty-four double tangents in their eight common tangent planes touch a common quadric K; the two associated quadrics H and K having, like the two E and F, the same common self-reciprocal tetrahedron with the original three S_1, S_2, S_3.*

4°. The equations, in point (or quadriplanar) coordinates, of the three original quadrics S_1, S_2, S_3 referred to the four faces of their common self-reciprocal tetrahedron,† being

$$\left.\begin{aligned} a_1\alpha^2 + b_1\beta^2 + c_1\gamma^2 + d_1\delta^2 = 0 \\ a_2\alpha^2 + b_2\beta^2 + c_2\gamma^2 + d_2\delta^2 = 0 \\ a_3\alpha^2 + b_3\beta^2 + c_3\gamma^2 + d_3\delta^2 = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1),$$

those, in similar coordinates to the same planes of reference, of the two associated quadrics E and F may be obtained immediately as follows.

As regards the equation of E. The three tangent planes L_1, L_2, L_3 at any one of the eight points $\alpha\beta\gamma\delta$ common to S_1, S_2, S_3, of which the equations are

$$\left.\begin{aligned} a_1\alpha.x + b_1\beta.y + c_1\gamma.z + d_1\delta.w = 0 \\ a_2\alpha.x + b_2\beta.y + c_2\gamma.z + d_2\delta.w = 0 \\ a_3\alpha.x + b_3\beta.y + c_3\gamma.z + d_3\delta.w = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2),$$

* It will appear in the sequel, that for the same system of three original quadrics S_1, S_2, S_3, there are four different quadrics H, and also four different quadrics K, touched by the two aforesaid systems of double tangents respectively; both quartets being related as above to the original triad.

† Their equations in plane (or tangential) coordinates, referred to the four vertices of the tetrahedron, would be precisely of the same form and for all the purposes of the present paper as easily manageable as the

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all touching the quadric E; if the equation of the latter be

$$a_e x^2 + b_e y^2 + c_e z^2 + d_e w^2 = 0 \quad \ldots\ldots\ldots\ldots (3),$$

we have at once, by comparison of coefficients, the three relations

$$\left.\begin{aligned} \frac{a_1^2}{a_e}\alpha^2 + \frac{b_1^2}{b_e}\beta^2 + \frac{c_1^2}{c_e}\gamma^2 + \frac{d_1^2}{d_e}\delta^2 = 0 \\ \frac{a_2^2}{a_e}\alpha^2 + \frac{b_2^2}{b_e}\beta^2 + \frac{c_2^2}{c_e}\gamma^2 + \frac{d_2^2}{d_e}\delta^2 = 0 \\ \frac{a_3^2}{a_e}\alpha^2 + \frac{b_3^2}{b_e}\beta^2 + \frac{c_3^2}{c_e}\gamma^2 + \frac{d_3^2}{d_e}\delta^2 = 0 \end{aligned}\right\} \ldots\ldots\ldots (4),$$

from which, equating the values of $\alpha^2 : \beta^2 : \gamma^2 : \delta^2$, as given by them, with those given by the equations (1), it follows at once that

$$a_e : b_e : c_e : d_e = \frac{(b_1 c_2 d_3)}{(b_1^2 c_2^2 d_3^2)} : \frac{(c_1 d_2 a_3)}{(c_1^2 d_2^2 a_3^2)} : \frac{(d_1 a_2 b_3)}{(d_1^2 a_2^2 b_3^2)} : \frac{(a_1 b_2 c_3)}{(a_1^2 b_2^2 c_3^2)} \quad \ldots\ldots\ldots\ldots (5),$$

the several quantities within parentheses representing determinants.

As regards the equation of F. The three points of contact P_1, P_2, P_3 of any one of the eight tangent planes common to S_1, S_2, S_3, the coordinates α_1, &c., α_2, &c., α_3, &c., of which, by virtue of the evident relations $a_1\alpha_1 = a_2\alpha_2 = a_3\alpha_3$, $b_1\beta_1 = b_2\beta_2 = b_3\beta_3$, $c_1\gamma_1 = c_2\gamma_2 = c_3\gamma_3$, $d_1\delta_1 = d_2\delta_2 = d_3\delta_3$, are given by the equations

$$\left.\begin{aligned} \frac{a_1^2}{a_1}\alpha_1^2 + \frac{b_1^2}{b_1}\beta_1^2 + \frac{c_1^2}{c_1}\gamma_1^2 + \frac{d_1^2}{d_1}\delta_1^2 = 0 \\ \frac{a_1^2}{a_2}\alpha_1^2 + \frac{b_1^2}{b_2}\beta_1^2 + \frac{c_1^2}{c_2}\gamma_1^2 + \frac{d_1^2}{d_2}\delta_1^2 = 0 \\ \frac{a_1^2}{a_3}\alpha_1^2 + \frac{b_1^2}{b_3}\beta_1^2 + \frac{c_1^2}{c_3}\gamma_1^2 + \frac{d_1^2}{d_3}\delta_1^2 = 0 \end{aligned}\right\} \ldots\ldots (6),$$

with two similar triads for $\alpha_2\beta_2\gamma_2\delta_2$ and $\alpha_3\beta_3\gamma_3\delta_3$, all lying on the quadric F; if the equation of the latter be

$$a_f x^2 + b_f y^2 + c_f z^2 + d_f w^2 = 0 \quad \ldots\ldots\ldots\ldots (7),$$

then, from the three relations resulting from the substitution in it for $x^2 : y^2 : z^2 : w^2$ of the values of $\alpha_1^2 : \beta_1^2 : \gamma_1^2 : \delta_1^2$,

above. With the use of those employed, however, the general reader will probably be more familiar.

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of $\alpha_2^2 : \beta_2^2 : \gamma_2^2 : \delta_2^2$, and of $\alpha_3^2 : \beta_3^2 : \gamma_3^2 : \delta_3^2$, as given by the three triads of equations (6), it follows at once that

$$a_f : b_f : c_f : d_f$$

$$= \frac{\left(\frac{1}{b_1^2}\,\frac{1}{c_2^2}\,\frac{1}{d_3^2}\right)}{\left(\frac{1}{b_1}\,\frac{1}{c_2}\,\frac{1}{d_3}\right)} : \frac{\left(\frac{1}{c_1^2}\,\frac{1}{d_2^2}\,\frac{1}{a_3^2}\right)}{\left(\frac{1}{c_1}\,\frac{1}{d_2}\,\frac{1}{a_3}\right)} : \frac{\left(\frac{1}{d_1^2}\,\frac{1}{a_2^2}\,\frac{1}{b_3^2}\right)}{\left(\frac{1}{d_1}\,\frac{1}{a_2}\,\frac{1}{b_3}\right)} : \frac{\left(\frac{1}{a_1^2}\,\frac{1}{b_2^2}\,\frac{1}{c_3^2}\right)}{\left(\frac{1}{a_1}\,\frac{1}{b_2}\,\frac{1}{c_3}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

the several quantities within parentheses, as before, representing determinants.*

The two quadrics E and F, whose equations (3) and (7) are thus known from the proportions (5) and (8), will be indeterminate, the former, as appears from (5), when the several determinants $(b_1c_2d_3)$, $(c_1d_2a_3)$, $(d_1a_2b_3)$, $(a_1b_2c_3)$, and the latter, as appears from (8), when the several determinants $\left(\frac{1}{b_1}\,\frac{1}{c_2}\,\frac{1}{d_3}\right)$, $\left(\frac{1}{c_1}\,\frac{1}{d_2}\,\frac{1}{a_3}\right)$, $\left(\frac{1}{d_1}\,\frac{1}{a_2}\,\frac{1}{b_3}\right)$, $\left(\frac{1}{a_1}\,\frac{1}{b_2}\,\frac{1}{c_3}\right)$, all vanish. Of this the reason is obvious, the three quadrics S_1, S_2, S_3 having a common curve of intersection, and therefore an infinite number of common points, in the former case, and a common developable of circumscription, and therefore an infinite number of common tangent planes, in the latter case.

5°. The two triads of equations, which give by elimination the values of the several ratios $a_k : b_k : c_k : d_k$ and $a_k : b_k : c_k : d_k$ for the two quadrics, or rather, as will presently appear, for the two quartets of quadrics, H and K, may also be readily obtained as follows.

As regards the triad for H. The three lines of intersection I_1, I_2, I_3 of the three pairs of tangent planes L_2 and L_3, L_3 and L_1, L_1 and L_2 of the preceding article, whose pairs

* For a triad of quadrics S_1', S_2', S_3', whose coefficients a_1', b_1', &c., are the reciprocals of those a_1, b_1, &c., of S_1, S_2, S_3, the coefficients a_s', &c., and a_f', &c., for E' and F' would, by the above proportions (5) and (8), be the reciprocals of a_f, &c., and a_s, &c., for F and E respectively. The reason of this is manifest, the three pairs of quadrics S_1 and S_1', S_2 and S_2', S_3 and S_3' being then reciprocal polars to each other with respect to the imaginary quadric $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$, and the two pairs E and F', E' and F being in consequence similarly related to each other. The same remark may, for the same reason, be applied generally to the several equations connected with the two quadrics, or rather with the two quartets of quadrics, H and K, as given in the succeeding articles, and might also, for a similar reason, be so applied, were the entire system of equations expressed in tangential instead of as above in quadriplanar coordinates.

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of equations are those of their determining pairs of planes (2), all touching the quadric H, if the equation of the latter be

$$a_4x^2 + b_4y^2 + c_4z^2 + d_4w^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (9),$$

then, eliminating between it and those of I_1, of I_2, and of I_3, successively, any two x and y of the four variables x, y, z, w, equating to nothing the discriminants of the three resulting quadratics for the ratios of the remaining two z and w, and substituting in the three results the values of the several ratios $\alpha^2 : \beta^2 : \gamma^2 : \delta^2$ as given by equations (1), we get, to determine the ratios of any three of the four coefficients a_4, b_4, c_4, d_4 to the fourth, the three equations

$$\left.\begin{aligned}
&(b_1c_2d_3)\,(a_1b_2c_3)\,(a_2d_3)^2.b_4c_4 + (c_1d_2a_3)\,(d_1a_2b_3)\,(b_2c_3)^2.a_4d_4 \\
&+ (c_1d_2a_3)\,(a_1b_2c_3)\,(b_2d_3)^2.c_4a_4 + (d_1a_2b_3)\,(b_1c_2d_3)\,(c_2a_3)^2.b_4d_4 \\
&+ (d_1a_2b_3)\,(a_1b_2c_3)\,(c_2d_3)^2.a_4b_4 + (b_1c_2d_3)\,(c_1d_2a_3)\,(a_2b_3)^2.c_4d_4 = 0, \\
&(b_1c_2d_3)\,(a_1b_2c_3)\,(a_3d_1)^2.b_4c_4 + (c_1d_2a_3)\,(d_1a_2b_3)\,(b_3c_1)^2.a_4d_4 \\
&+ (c_1d_2a_3)\,(a_1b_2c_3)\,(b_3d_1)^2.c_4a_4 + (d_1a_2b_3)\,(b_1c_2d_3)\,(c_3a_1)^2.b_4d_4 \\
&+ (d_1a_2b_3)\,(a_1b_2c_3)\,(c_3d_1)^2.a_4b_4 + (b_1c_2d_3)\,(c_1d_2a_3)\,(a_3b_1)^2.c_4d_4 = 0, \\
&(b_1c_2d_3)\,(a_1b_2c_3)\,(a_1d_2)^2.b_4c_4 + (c_1d_2a_3)\,(d_1a_2b_3)\,(b_1c_2)^2.a_4d_4 \\
&+ (c_1d_2a_3)\,(a_1b_2c_3)\,(b_1d_2)^2.c_4a_4 + (d_1a_2b_3)\,(b_1c_2d_3)\,(c_1a_3)^2.b_4d_4 \\
&+ (d_1a_2b_3)\,(a_1b_2c_3)\,(c_1d_2)^2.a_4b_4 + (b_1c_2d_3)\,(c_1d_2a_3)\,(a_1b_3)^2.c_4d_4 = 0,
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (10),$$

the elimination between which of any two of the three ratios leads to an equation of the fourth degree for the third; thus showing, as observed above, that there are four different quadrics H which possess the property in question of touching the eight triads of lines I_1, I_2, I_3, &c.

As regards the triad for K. The three lines of connection J_1, J_2, J_3 of the three pairs of points of contact P_2 and P_3, P_3 and P_1, P_1 and P_2 of the preceding article, whose coordinates α_1, &c., α_2, &c., α_3, &c., as there shewn, are given by the three triads of equations (6), all touching the quadric K; if the equation of the latter be

$$a_4x^2 + b_4y^2 + c_4z^2 + d_4w^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (11),$$

then, substituting successively in it for x, &c., $\alpha_2 + \lambda_1\alpha_3$, &c., $\alpha_3 + \lambda_2\alpha_1$, &c., $\alpha_1 + \lambda_3\alpha_2$, &c., equating to nothing the discriminants of the three resulting quadratics for λ_1, λ_2, λ_3, and substituting in the three results the values of the three groups of ratios

$$\alpha_1^2 : \beta_1^2 : \gamma_1^2 : \delta_1^2, \quad \alpha_2^2 : \beta_2^2 : \gamma_2^2 : \delta_2^2, \quad \alpha_3^2 : \beta_3^2 : \gamma_3^2 : \delta_3^2,$$

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as given by the three triads of equations (6), we get to determine the ratios of any three of the four coefficients a_k, b_k, c_k, d_k to the fourth, the three equations

$$\left.\begin{aligned}
&\left(\frac{1}{c_1}\frac{1}{d_2}\frac{1}{a_3}\right)\left(\frac{1}{d_1}\frac{1}{a_2}\frac{1}{b_3}\right)\left(\frac{1}{b_2}\frac{1}{c_3}\right)^2.b_kc_k+\left(\frac{1}{b_1}\frac{1}{c_2}\frac{1}{d_3}\right)\left(\frac{1}{a_1}\frac{1}{b_2}\frac{1}{c_3}\right)\left(\frac{1}{a_2}\frac{1}{d_3}\right)^2.a_kd_k\\
&+\left(\frac{1}{d_1}\frac{1}{a_2}\frac{1}{b_3}\right)\left(\frac{1}{b_1}\frac{1}{c_2}\frac{1}{d_3}\right)\left(\frac{1}{c_2}\frac{1}{a_3}\right)^2.c_ka_k+\left(\frac{1}{c_1}\frac{1}{d_2}\frac{1}{a_3}\right)\left(\frac{1}{a_1}\frac{1}{b_2}\frac{1}{c_3}\right)\left(\frac{1}{b_2}\frac{1}{d_3}\right)^2.b_kd_k\\
&+\left(\frac{1}{b_1}\frac{1}{c_2}\frac{1}{d_3}\right)\left(\frac{1}{c_1}\frac{1}{d_2}\frac{1}{a_3}\right)\left(\frac{1}{a_2}\frac{1}{b_3}\right)^2.a_kb_k+\left(\frac{1}{d_1}\frac{1}{a_2}\frac{1}{b_3}\right)\left(\frac{1}{a_1}\frac{1}{b_2}\frac{1}{c_3}\right)\left(\frac{1}{c_2}\frac{1}{d_3}\right)^2.c_kd_k=0,\\
&\left(\frac{1}{c_1}\frac{1}{d_2}\frac{1}{a_3}\right)\left(\frac{1}{d_1}\frac{1}{a_2}\frac{1}{c_3}\right)\left(\frac{1}{b_3}\frac{1}{c_1}\right)^2.b_kc_k+\left(\frac{1}{b_1}\frac{1}{c_2}\frac{1}{d_3}\right)\left(\frac{1}{a_1}\frac{1}{b_2}\frac{1}{c_3}\right)\left(\frac{1}{a_2}\frac{1}{d_1}\right)^2.a_kd_k\\
&+\left(\frac{1}{d_1}\frac{1}{a_2}\frac{1}{b_3}\right)\left(\frac{1}{b_1}\frac{1}{c_2}\frac{1}{d_3}\right)\left(\frac{1}{c_3}\frac{1}{a_1}\right)^2.c_ka_k+\left(\frac{1}{c_1}\frac{1}{d_2}\frac{1}{a_3}\right)\left(\frac{1}{a_1}\frac{1}{b_2}\frac{1}{c_3}\right)\left(\frac{1}{b_3}\frac{1}{d_1}\right)^2.b_kd_k\\
&+\left(\frac{1}{b_1}\frac{1}{c_2}\frac{1}{d_3}\right)\left(\frac{1}{c_1}\frac{1}{d_2}\frac{1}{a_3}\right)\left(\frac{1}{a_3}\frac{1}{b_1}\right)^2.a_kb_k+\left(\frac{1}{d_1}\frac{1}{a_2}\frac{1}{b_3}\right)\left(\frac{1}{a_1}\frac{1}{b_2}\frac{1}{c_3}\right)\left(\frac{1}{c_3}\frac{1}{d_1}\right)^2.c_kd_k=0,\\
&\left(\frac{1}{c_1}\frac{1}{d_2}\frac{1}{a_3}\right)\left(\frac{1}{d_1}\frac{1}{a_2}\frac{1}{b_3}\right)\left(\frac{1}{b_1}\frac{1}{c_2}\right)^2.b_kc_k+\left(\frac{1}{b_1}\frac{1}{c_2}\frac{1}{d_3}\right)\left(\frac{1}{a_1}\frac{1}{b_2}\frac{1}{c_3}\right)\left(\frac{1}{a_1}\frac{1}{d_2}\right)^2.a_kd_k\\
&+\left(\frac{1}{d_1}\frac{1}{a_2}\frac{1}{b_3}\right)\left(\frac{1}{b_1}\frac{1}{c_2}\frac{1}{d_3}\right)\left(\frac{1}{c_1}\frac{1}{a_2}\right)^2.c_ka_k+\left(\frac{1}{c_1}\frac{1}{d_2}\frac{1}{a_3}\right)\left(\frac{1}{a_1}\frac{1}{b_2}\frac{1}{c_3}\right)\left(\frac{1}{b_1}\frac{1}{d_2}\right)^2.b_kd_k\\
&+\left(\frac{1}{b_1}\frac{1}{c_2}\frac{1}{d_3}\right)\left(\frac{1}{c_1}\frac{1}{d_2}\frac{1}{a_3}\right)\left(\frac{1}{a_1}\frac{1}{b_2}\right)^2.a_kb_k+\left(\frac{1}{d_1}\frac{1}{a_2}\frac{1}{b_3}\right)\left(\frac{1}{a_1}\frac{1}{b_2}\frac{1}{c_3}\right)\left(\frac{1}{c_1}\frac{1}{d_2}\right)^2.c_kd_k=0,
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

the elimination between which of any two of the three ratios leads, as for the case of H, to an equation of the fourth degree for the third; thus shewing, as observed above, that there are also four different quadrics K which possess the property in question of touching the eight triads of lines J_1, J_2, J_3, &c.*

The two quartets of quadrics H and K, whose equations (9) and (11) are those implicitly given by the two triads (10) and (12), will be indeterminate, as appears from the latter, under the same circumstances as the two E and F;

* Of the two different triads of equations (10) and (12), obtained above by distinct processes, either, by virtue of the general observation in the note to Art. 4°, might be written down at once from the other: the two associated quartets of quadrics H and K', or K and H', for two reciprocal triads of original quadrics S_1, S_2, S_3 and S_1', S_2', S_3', being manifestly reciprocal to each other.

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the reason, see preceding article, being of course the same in both cases.

6°. In Geometry of Two Dimensions, for any two conics S_1, S_2 having a common self-reciprocal triangle (that is for any two conics whatever), every line L whose chords of intersection with both cut each other harmonically touches their associated conic E, and every point P whose angles of subtense with both cut each other harmonically lies on their associated conic F. See Salmon's *Conic Sections*, Edit. 5, Arts. 334 and 335.

Analogously, in Geometry of Three Dimensions, for any three quadrics S_1, S_2, S_3 having a common self-reciprocal tetrahedron, every line I whose chords of intersection with every two of them cut each other harmonically touches their associated quartet of quadrics H, and every line J whose angles of subtense with every two of them cut each other harmonically touches their associated quartet of quadrics K.

As regards the former property. If $\alpha'\beta'\gamma'\delta'$ and $\alpha''\beta''\gamma''\delta''$ be the coordinates of any two points P', P'' on any one of the lines I in question; then, in order that the line $P'P''$ or I should intersect S_2 and S_3, S_3 and S_1, S_1 and S_2, respectively, in pairs of chords cutting each other harmonically, we must have the three equations of condition*

$$\left.\begin{aligned}
&(b_2c_3 + b_3c_2)(\beta'\gamma'' - \beta''\gamma')^2 + (a_2d_3 + a_3d_2)(\alpha'\delta'' - \alpha''\delta')^2 \\
&+ (c_2a_3 + c_3a_2)(\gamma'\alpha'' - \gamma''\alpha')^2 + (b_2d_3 + b_3d_2)(\beta'\delta'' - \beta''\delta')^2 \\
&+ (a_2b_3 + a_3b_2)(\alpha'\beta'' - \alpha''\beta')^2 + (c_2d_3 + c_3d_2)(\gamma'\delta'' - \gamma''\delta')^2 = 0, \\
&(b_3c_1 + b_1c_3)(\beta'\gamma'' - \beta''\gamma')^2 + (a_3d_1 + a_1d_3)(\alpha'\delta'' - \alpha''\delta')^2 \\
&+ (c_3a_1 + c_1a_3)(\gamma'\alpha'' - \gamma''\alpha')^2 + (b_3d_1 + b_1d_3)(\beta'\delta'' - \beta''\delta')^2 \\
&+ (a_3b_1 + a_1b_3)(\alpha'\beta'' - \alpha''\beta')^2 + (c_3d_1 + c_1d_3)(\gamma'\delta'' - \gamma''\delta')^2 = 0, \\
&(b_1c_2 + b_2c_1)(\beta'\gamma'' - \beta''\gamma')^2 + (a_1d_2 + a_2d_1)(\alpha'\delta'' - \alpha''\delta')^2 \\
&+ (c_1a_2 + c_2a_1)(\gamma'\alpha'' - \gamma''\alpha')^2 + (b_1d_2 + b_2d_1)(\beta'\delta'' - \beta''\delta')^2 \\
&+ (a_1b_2 + a_2b_1)(\alpha'\beta'' - \alpha''\beta')^2 + (c_1d_2 + c_2d_1)(\gamma'\delta'' - \gamma''\delta')^2 = 0,
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(13),$$

* See Salmon's *Conic Sections*, 5th edit., p. 295, note. Where the two conics U and V may evidently be replaced by two quadrics, and the two polar lines P and Q with respect to them by the two corresponding polar planes with respect to the quadrics.

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and, in order that it should in addition touch a quadric M having the same common self-reciprocal tetrahedron with S_1, S_2, S_3, if the equation of the latter be

$$a_m x^2 + b_m y^2 + c_m z^2 + d_m w^2 = 0 \quad \ldots\ldots\ldots\ldots (14),$$

we must have the additional equation of condition

$$\left.\begin{aligned} &b_m c_m (\beta'\gamma'' - \beta''\gamma')^2 + c_m a_m (\gamma'\alpha'' - \gamma''\alpha')^2 + a_m b_m (\alpha'\beta'' - \alpha''\beta')^2 \\ &+ a_m d_m (\alpha'\delta'' - \alpha''\delta')^2 + b_m d_m (\beta'\delta'' - \beta''\delta')^2 + c_m d_m (\gamma'\delta'' - \gamma''\delta')^2 = 0 \end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (15),$$

between these four equations of condition (13) and (15) eliminating any three of the six variables $(\beta'\gamma'' - \beta''\gamma')^2$, &c., which enter into them all linearly, and equating to nothing in the result the coefficients of the remaining three, we get three* equations of the same form as those (10) of the preceding article for the determination of the three ratios of any three of the four coefficients a_m, b_m, c_m, d_m to the fourth; thus shewing that there are four different quadrics M touched by the system of lines I, which (as will be presently shewn) are in fact no other than the four H that are touched by the eight triads of lines I_1, I_2, I_3, &c., of the preceding article, the whole twenty-four of which are manifestly included in the system I.

As regards the latter property. If $p'q'r's'$ and $p''q''r''s''$ be the coefficients of any two planes L', L'' through any one of the lines J in question; then, in order that the line $L'L''$ or J should subtend S_2 and S_3, S_3 and S_1, S_1 and S_2, respectively, in pairs of angles cutting each other harmonically, we must have the three equations of condition,†

* Denoting by $l_1, m_1, n_1, p_1, q_1, r_1$; l_2, &c., the eighteen coefficients of the six variables $(\beta'\gamma'' - \beta''\gamma')^2$, &c., in equations (13), and by $l_x, m_x, n_x, p_x, q_x, r_x$ the six of the same in equation (15); the above three equations of connection between the entire twenty-four, as well as all the other triads which could be obtained by the same process, may evidently be exhibited together in the determinental form

$$\begin{vmatrix} l_1, m_1, n_1, p_1, q_1, r_1 \\ l_2, m_2, n_2, p_2, q_2, r_2 \\ l_3, m_3, n_3, p_3, q_3, r_3 \\ l_x, m_x, n_x, p_x, q_x, r_x \end{vmatrix} = 0.$$

† These may either be obtained directly by Dr. Salmon's method, like the preceding conditions (13), or they may be written down at once from the latter by substituting for the two planes L', L'' their

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$$\left.\begin{aligned}
&\left(\frac{1}{b_2c_2}+\frac{1}{b_3c_3}\right)(q'r''-q''r')^2+\left(\frac{1}{a_2d_2}+\frac{1}{a_3d_3}\right)(p's''-p''s')^2\\
&+\left(\frac{1}{c_2a_2}+\frac{1}{c_3a_3}\right)(r'p''-r''p')^2+\left(\frac{1}{b_2d_2}+\frac{1}{b_3d_3}\right)(q's''-q''s')^2\\
&+\left(\frac{1}{a_2b_2}+\frac{1}{a_3b_3}\right)(p'q''-p''q')^2+\left(\frac{1}{c_2d_2}+\frac{1}{c_3d_3}\right)(r's''-r''s')^2=0,\\
&\left(\frac{1}{b_3c_1}+\frac{1}{b_1c_3}\right)(q'r''-q''r')^2+\left(\frac{1}{a_3d_1}+\frac{1}{a_1d_3}\right)(p's''-p''s')^2\\
&+\left(\frac{1}{c_3a_1}+\frac{1}{c_1a_3}\right)(r'p''-r''p')^2+\left(\frac{1}{b_3d_1}+\frac{1}{b_1d_3}\right)(q's''-q''s')^2\\
&+\left(\frac{1}{a_3b_1}+\frac{1}{a_1b_3}\right)(p'q''-p''q')^2+\left(\frac{1}{c_3d_1}+\frac{1}{c_1d_3}\right)(r's''-r''s')^2=0,\\
&\left(\frac{1}{b_1c_2}+\frac{1}{b_2c_1}\right)(q'r''-q''r')^2+\left(\frac{1}{a_1d_2}+\frac{1}{a_2d_1}\right)(p's''-p''s')^2\\
&+\left(\frac{1}{c_1a_2}+\frac{1}{c_2a_1}\right)(r'p''-r''p')^2+\left(\frac{1}{b_1d_2}+\frac{1}{b_2d_1}\right)(q's''-q''s')^2\\
&+\left(\frac{1}{a_1b_2}+\frac{1}{a_2b_1}\right)(p'q''-p''q')^2+\left(\frac{1}{c_1d_2}+\frac{1}{c_2d_1}\right)(r's''-r''s')^2=0,
\end{aligned}\right\}$$

.....(16),

and, in order that it should in addition touch a quadric N having the same common self-reciprocal tetrahedron with S_1, S_2, S_3, if the equation of the latter be

$$a_nx^2+b_ny^2+c_nz^2+d_nw^2=0 \text{ ..... } (17),$$

we must have the additional equation of condition

$$\left.\begin{aligned}
&\frac{1}{b_nc_n}(q'r''-q''r')^2+\frac{1}{c_na_n}(r'p''-r''p')^2+\frac{1}{a_nb_n}(p'q''-p''q')^2\\
&+\frac{1}{a_nd_n}(p's''-p''s')^2+\frac{1}{b_nd_n}(q's''-q''s')^2+\frac{1}{c_nd_n}(r's''-r''s')^2=0
\end{aligned}\right\}$$

..... (18),

two poles P', P'', for the line $L'L''$ its polar $P'P''$, and for the three quadrics S_1, S_2, S_3 their three reciprocals S_1', S_2', S_3', with respect to the imaginary quadric $\alpha^2+\beta^2+\gamma^2+\delta^2=0$. See Art. 4°, Note.

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between these four equations of condition 16 and 18' eliminating any three of the six variables $q'r'' - q''r'$, &c., which, as in the former case, enter into them all linearly, and equating to nothing in the result the coefficients of the remaining three, we get three* equations of the same form as those 12 of the preceding article for the determination of the three ratios of any three of the four coefficients a_4, b_4, c_4, d_4 to the fourth: thus shewing that there are four different quadrics N touched by the system of lines J, which as will be presently shewn are in fact no other than the four K that are touched by the eight triads of lines J_1, J_2, J_3, &c. of the preceding article, the whole twenty-four of which are manifestly included in the system J.

7°. That the two quartets of quadrics M and N are in fact no other than the two H and K, though evident indirectly from the two systems of lines I and J including, as they manifestly do respectively, the two systems I_1, I_2, I_3, &c., and J_1, J_2, J_3, &c., of Art. 5°, may easily be shewn directly as follows:

As regards the first identity. The three lines of intersection I_1, I_2, I_3 (Art. 5°) of the three pairs of planes L_2 and L_3, L_3 and L_1, L_1 and L_2 (Art. 4°) all touching the quadric H, the equation of the cone which from their common origin $\alpha\beta\gamma\delta$ as vertex envelopes that quadric must consequently be of the form

$$k_1 . L_2L_3 + k_2 . L_3L_1 + k_3 . L_1L_2 = 0 \ldots\ldots\ldots\ldots (19).$$

Substituting in this for L_1, L_2, L_3 from equations (2) Art. 4°, and identifying in the result the coefficients of the six products xy, &c., with those of the same in the known equation of the same cone, viz.:

$$\left.\begin{array}{l} b_4c_4(\gamma y - \beta z)^2 + c_4a_4(\alpha z - \gamma x)^2 + a_4b_4(\beta x - \alpha y)^2 \\ + a_4d_4(\delta x - \alpha w)^2 + b_4d_4(\delta y - \beta w)^2 + c_4d_4(\delta z - \gamma w)^2 = 0 \end{array}\right\} \ldots (20),$$

we get, to determine the three multipliers k_1, k_2, k_3, and the

* Denoting, as in the note to the preceding property, by l_1, &c., the twenty-four coefficients of the six variables $(q'r'' - q''r')^2$, &c., in the four equations of condition (16) and (18); the above three equations of connection between them, as well as all the other triads which could be obtained by the same process, may evidently be exhibited together in the same determinantal form there given.

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three ratios of any three of the four coefficients a_h, b_h, c_h, d_h to the fourth, the six* equations

$$\left.\begin{aligned}
&k_1(b_2c_3+b_3c_2)+k_2(b_3c_1+b_1c_3)+k_3(b_1c_2+b_2c_1)=-2b_hc_h\\
&k_1(c_2a_3+c_3a_2)+k_2(c_3a_1+c_1a_3)+k_3(c_1a_2+c_2a_1)=-2c_ha_h\\
&k_1(a_2b_3+a_3b_2)+k_2(a_3b_1+a_1b_3)+k_3(a_1b_2+a_2b_1)=-2a_hb_h\\
&k_1(a_2d_3+a_3d_2)+k_2(a_3d_1+a_1d_3)+k_3(a_1d_2+a_2d_1)=-2a_hd_h\\
&k_1(b_2d_3+b_3d_2)+k_2(b_3d_1+b_1d_3)+k_3(b_1d_2+b_2d_1)=-2b_hd_h\\
&k_1(c_2d_3+c_3d_2)+k_2(c_3d_1+c_1d_3)+k_3(c_1d_2+c_2d_1)=-2c_hd_h
\end{aligned}\right\}\quad\ldots(21),$$

the elimination between which (in any of the fifteen different ways in which it may be effected by mere inspection) of the three multipliers gives a triad of equations of the same form with, and of course also equivalent to, the three (10) of Art. 5° for the determination of the three ratios.

Multiplying now the three equations (13) and the equation (15) of the preceding article (6°) by k_1, k_2, k_3, and 2, respectively, adding the whole four so multiplied together, and equating to nothing in the result the coefficients of the six variables $(\beta'\gamma''-\beta''\gamma')^2$, &c., we get precisely the same system of equations (21) for the determination of the three multipliers k_1, k_2, k_3, and of the three ratios of any three of the four coefficients a_m, b_m, c_m, d_m to the fourth; thus establishing the identity of the two quadrics, or rather of the two quartets of quadrics, *H* and *M*.

As regards the second identity. The three lines of connection J_1, J_2, J_3 (Art. 5°) of the three pairs of points P_2 and P_3, P_3 and P_1, P_1 and P_2 (Art. 4°) all touching the quadric K, the equation of the cone which envelopes that quadric along its section by their plane must consequently be of the form

$$k_1.Q_2Q_3+k_2.Q_3Q_1+k_3.Q_1Q_2=0\ldots\ldots\ldots(22),$$

where $Q_1=0$, $Q_2=0$, $Q_3=0$ are those of the polar planes of P_1, P_2, P_3 respectively with respect to it. Substituting,

* By virtue of these six the remaining four, got by identifying the coefficients of the four squares x^2, &c. in the two equations of the cone, are easily seen to be satisfied identically. As, of course, ought to be the case, the assumptions leading apparently to ten equations for the determination of six magnitudes being, as already shewn, consistent.

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as above, in this for Q_1, &c., $a_k\alpha_1.x+$&c., $a_k\alpha_2.x+$&c., $a_k\alpha_3.x+$&c., and identifying, as before, in the result the coefficients of the six products xy, &c., with those of the same in the known equation of the same cone, viz.

$$\left.\begin{aligned} &b_kc_k(\gamma y-\beta z)^2+c_ka_k(\alpha z-\gamma x)^2+a_kb_k(\beta x-\alpha y)^2 \\ &+a_kd_k(\delta x-\alpha w)^2+b_kd_k(\delta y-\beta w)^2+c_kd_k(\delta z-\gamma w)^2=0 \end{aligned}\right\}\ldots(23),$$

where $\alpha\beta\gamma\delta$ is the pole of $P_1P_2P_3$, we get, precisely as in the preceding case, to determine the three multipliers k_1, k_2, k_3, and the three ratios of any three of the four coefficients a_k, b_k, c_k, d_k to the fourth, the six* equations

$$\left.\begin{aligned}
k_1\left(\frac{1}{b_2c_3}+\frac{1}{b_3c_2}\right)+k_2\left(\frac{1}{b_3c_1}+\frac{1}{b_1c_3}\right)+k_3\left(\frac{1}{b_1c_2}+\frac{1}{b_2c_1}\right)&=-\frac{2}{b_kc_k}\\
k_1\left(\frac{1}{c_2a_3}+\frac{1}{c_3a_2}\right)+k_2\left(\frac{1}{c_3a_1}+\frac{1}{c_1a_3}\right)+k_3\left(\frac{1}{c_1a_2}+\frac{1}{c_2a_1}\right)&=-\frac{2}{c_ka_k}\\
k_1\left(\frac{1}{a_2b_3}+\frac{1}{a_3b_2}\right)+k_2\left(\frac{1}{a_3b_1}+\frac{1}{a_1b_3}\right)+k_3\left(\frac{1}{a_1b_2}+\frac{1}{a_2b_1}\right)&=-\frac{2}{a_kb_k}\\
k_1\left(\frac{1}{a_2d_3}+\frac{1}{a_3d_2}\right)+k_2\left(\frac{1}{a_3d_1}+\frac{1}{a_1d_3}\right)+k_3\left(\frac{1}{a_1d_2}+\frac{1}{a_2d_1}\right)&=-\frac{2}{a_kd_k}\\
k_1\left(\frac{1}{b_2d_3}+\frac{1}{b_3d_2}\right)+k_2\left(\frac{1}{b_3d_1}+\frac{1}{b_1d_3}\right)+k_3\left(\frac{1}{b_1d_2}+\frac{1}{b_2d_1}\right)&=-\frac{2}{b_kd_k}\\
k_1\left(\frac{1}{c_2d_3}+\frac{1}{c_3d_2}\right)+k_2\left(\frac{1}{c_3d_1}+\frac{1}{c_1d_3}\right)+k_3\left(\frac{1}{c_1d_2}+\frac{1}{c_2d_1}\right)&=-\frac{2}{c_kd_k}
\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(24),$$

the elimination between which, as before, of the three multipliers gives a triad of equations of the same form with, and of course also equivalent to, the three (12) of Art. 5° for the determination of the three ratios.

Multiplying now the three equations (16) and the equation (18) of the preceding article (6°) by k_1, k_2, k_3, and 2, respectively, adding the whole four so multiplied together, and equating to nothing in the result the coefficients of the six variables $(q'r''-q''r')^2$, &c., we get precisely the same system of equations (24) for the determination of the three multipliers k_1, k_2, k_3, and of the ratios of any three of the four coefficients a_n, b_n, c_n, d_n to the fourth; thus establishing

* See preceding note, which may in fact be repeated here.

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the identity of the two quadrics, or rather of the two quartets of quadrics, K and N.

8°. The relations connecting the four quartets of coefficients a_1, &c., a_2, &c., a_3, &c., a_4, &c., of four quadrics S_1, S_2, S_3, S_4, having a common self-reciprocal tetrahedron, which, like the two reciprocal quartets H and K of the three preceding articles, are such that every line touching any three of them touches the fourth also, may be readily obtained as follows.

The equations of the four quadrics being

$$\left.\begin{aligned} a_1\alpha^2 + b_1\beta^2 + c_1\gamma^2 + d_1\delta^2 = 0 \\ a_2\alpha^2 + b_2\beta^2 + c_2\gamma^2 + d_2\delta^2 = 0 \\ a_3\alpha^2 + b_3\beta^2 + c_3\gamma^2 + d_3\delta^2 = 0 \\ a_4\alpha^2 + b_4\beta^2 + c_4\gamma^2 + d_4\delta^2 = 0 \end{aligned}\right\} \ldots\ldots\ldots (25),$$

if $\alpha'\beta'\gamma'\delta'$ and $\alpha''\beta''\gamma''\delta''$ be the coordinates of any two arbitrary points P' and P'' on any line I touching any three of them, and therefore by hypothesis the fourth also, then, denoting by u, v, w, x, y, z the six variables $(\beta'\gamma'' - \gamma'\beta'')$, &c., of Art. 6°, we have, between them and the separate coefficients of the four quadrics, the four equations of condition

$$\left.\begin{aligned} b_1c_1.u^2 + c_1a_1.v^2 + a_1b_1.w^2 + a_1d_1.x^2 + b_1d_1.y^2 + c_1d_1.z^2 = 0 \\ b_2c_2.u^2 + c_2a_2.v^2 + a_2b_2.w^2 + a_2d_2.x^2 + b_2d_2.y^2 + c_2d_2.z^2 = 0 \\ b_3c_3.u^2 + c_3a_3.v^2 + a_3b_3.w^2 + a_3d_3.x^2 + b_3d_3.y^2 + c_3d_3.z^2 = 0 \\ b_4c_4.u^2 + c_4a_4.v^2 + a_4b_4.w^2 + a_4d_4.x^2 + b_4d_4.y^2 + c_4d_4.z^2 = 0 \end{aligned}\right\} \ldots(26),$$

which, holding simultaneously for all values of the six variables involved, give consequently, as the required relations between the whole sixteen coefficients, the system of fifteen equations

$$\begin{vmatrix} b_1c_1, & c_1a_1, & a_1b_1, & a_1d_1, & b_1d_1, & c_1d_1, \\ b_2c_2, & c_2a_2, & a_2b_2, & a_2d_2, & b_2d_2, & c_2d_2, \\ b_3c_3, & c_3a_3, & a_3b_3, & a_3d_3, & b_3d_3, & c_3d_3, \\ b_4c_4, & c_4a_4, & a_4b_4, & a_4d_4, & b_4d_4, & c_4d_4, \end{vmatrix} = 0 \ldots\ldots (27),$$

any three of which, however, that contain between them the

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six columns, and are consequently independent, involve, as is evident, the remaining twelve.

To exhibit, as may be done in a variety of ways, any independent triad of the above equations of connection in a symmetrical form; adding together the four equations of condition (25), from which they have been obtained, multiplied respectively by k_1, k_2, k_3, k_4, equating to nothing in the sum the separate coefficients of the six variables a^2, c^2, &c., and solving for the ratios of the four multipliers, firstly from the four triads containing a common letter, and secondly from the four residual triads, of the six resulting equations, we get, from the first quartet of triads, the following symmetrical groups of proportions:

$$\left.\begin{aligned}
a_1k_1 : a_2k_2 : a_3k_3 : a_4k_4 &= (b_2c_3d_4 : b_3c_4d_1) : (b_4c_1d_2 : b_1c_2d_3)\\
b_1k_1 : b_2k_2 : b_3k_3 : b_4k_4 &= (c_2d_3a_4 : c_3d_4a_1) : (c_4d_1a_2 : c_1d_2a_3)\\
c_1k_1 : c_2k_2 : c_3k_3 : c_4k_4 &= (d_2a_3b_4 : d_3a_4b_1) : (d_4a_1b_2 : d_1a_2b_3)\\
d_1k_1 : d_2k_2 : d_3k_3 : d_4k_4 &= (a_2b_3c_4 : a_3b_4c_1) : (a_4b_1c_2 : a_1b_2c_3)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28),$$

and from the second, the different but also symmetrical groups

$$\left.\begin{aligned}
b_1c_1d_1k_1 : b_2c_2d_2k_2 : b_3c_3d_3k_3 : b_4c_4d_4k_4 &\\
= \left(\frac{1}{b_2}\ \frac{1}{c_3}\ \frac{1}{d_4}\right) : \left(\frac{1}{b_3}\ \frac{1}{c_4}\ \frac{1}{d_1}\right) : \left(\frac{1}{b_4}\ \frac{1}{c_1}\ \frac{1}{d_2}\right) : \left(\frac{1}{b_1}\ \frac{1}{c_2}\ \frac{1}{d_3}\right)&\\
c_1d_1a_1k_1 : c_2d_2a_2k_2 : c_3d_3a_3k_3 : c_4d_4a_4k_4 &\\
= \left(\frac{1}{c_2}\ \frac{1}{d_3}\ \frac{1}{a_4}\right) : \left(\frac{1}{c_3}\ \frac{1}{d_4}\ \frac{1}{a_1}\right) : \left(\frac{1}{c_4}\ \frac{1}{d_1}\ \frac{1}{a_2}\right) : \left(\frac{1}{c_1}\ \frac{1}{d_2}\ \frac{1}{a_3}\right)&\\
d_1a_1b_1k_1 : d_2a_2b_2k_2 : d_3a_3b_3k_3 : d_4a_4b_4k_4 &\\
= \left(\frac{1}{d_2}\ \frac{1}{a_3}\ \frac{1}{b_4}\right) : \left(\frac{1}{d_3}\ \frac{1}{a_4}\ \frac{1}{b_1}\right) : \left(\frac{1}{d_4}\ \frac{1}{a_1}\ \frac{1}{b_2}\right) : \left(\frac{1}{d_1}\ \frac{1}{a_2}\ \frac{1}{b_3}\right)&\\
a_1b_1c_1k_1 : a_2b_2c_2k_2 : a_3b_3c_3k_3 : a_4b_4c_4k_4 &\\
= \left(\frac{1}{a_2}\ \frac{1}{b_3}\ \frac{1}{c_4}\right) : \left(\frac{1}{a_3}\ \frac{1}{b_4}\ \frac{1}{c_1}\right) : \left(\frac{1}{a_4}\ \frac{1}{b_1}\ \frac{1}{c_2}\right) : \left(\frac{1}{a_1}\ \frac{1}{b_2}\ \frac{1}{c_3}\right)&
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29),$$

from which two groups, by division of corresponding members,

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the four following symmetrical and separately independent triads of the required equations of condition

$$\left.\begin{aligned}
&\frac{b_1c_1d_1}{a_1}\cdot\frac{(b_2c_3d_4)}{\left(\frac{1}{b_2}\,\frac{1}{c_3}\,\frac{1}{d_4}\right)}=\frac{b_2c_2d_2}{a_2}\cdot\frac{(b_3c_4d_1)}{\left(\frac{1}{b_3}\,\frac{1}{c_4}\,\frac{1}{d_1}\right)}=\frac{b_3c_3d_3}{a_3}\cdot\frac{(b_4c_1d_2)}{\left(\frac{1}{b_4}\,\frac{1}{c_1}\,\frac{1}{d_2}\right)}\\
&\qquad\qquad=\frac{b_4c_4d_4}{a_4}\cdot\frac{(b_1c_2d_3)}{\left(\frac{1}{b_1}\,\frac{1}{c_2}\,\frac{1}{d_3}\right)}\\
&\frac{c_1d_1a_1}{b_1}\cdot\frac{(c_2d_3a_4)}{\left(\frac{1}{c_2}\,\frac{1}{d_3}\,\frac{1}{a_4}\right)}=\frac{c_2d_2a_2}{b_2}\cdot\frac{(c_3d_4a_1)}{\left(\frac{1}{c_3}\,\frac{1}{d_4}\,\frac{1}{a_1}\right)}=\frac{c_3d_3a_3}{b_3}\cdot\frac{(c_4d_1a_2)}{\left(\frac{1}{c_4}\,\frac{1}{d_1}\,\frac{1}{a_2}\right)}\\
&\qquad\qquad=\frac{c_4d_4a_4}{b_4}\cdot\frac{(c_1d_2a_3)}{\left(\frac{1}{c_1}\,\frac{1}{d_2}\,\frac{1}{a_3}\right)}\\
&\frac{d_1a_1b_1}{c_1}\cdot\frac{(d_2a_3b_4)}{\left(\frac{1}{d_2}\,\frac{1}{a_3}\,\frac{1}{b_4}\right)}=\frac{d_2a_2b_2}{c_2}\cdot\frac{(d_3a_4b_1)}{\left(\frac{1}{d_3}\,\frac{1}{a_4}\,\frac{1}{b_1}\right)}=\frac{d_3a_3b_3}{c_3}\cdot\frac{(d_4a_1b_2)}{\left(\frac{1}{d_4}\,\frac{1}{a_1}\,\frac{1}{b_2}\right)}\\
&\qquad\qquad=\frac{d_4a_4b_4}{c_4}\cdot\frac{(d_1a_2b_3)}{\left(\frac{1}{d_1}\,\frac{1}{a_2}\,\frac{1}{b_3}\right)}\\
&\frac{a_1b_1c_1}{d_1}\cdot\frac{(a_2b_3c_4)}{\left(\frac{1}{a_2}\,\frac{1}{b_3}\,\frac{1}{c_4}\right)}=\frac{a_2b_2c_2}{d_2}\cdot\frac{(a_3b_4c_1)}{\left(\frac{1}{a_3}\,\frac{1}{b_4}\,\frac{1}{c_1}\right)}=\frac{a_3b_3c_3}{d_3}\cdot\frac{(a_4b_1c_2)}{\left(\frac{1}{a_4}\,\frac{1}{b_1}\,\frac{1}{c_2}\right)}\\
&\qquad\qquad=\frac{a_4b_4c_4}{d_4}\cdot\frac{(a_1b_2c_3)}{\left(\frac{1}{a_1}\,\frac{1}{b_2}\,\frac{1}{c_3}\right)}
\end{aligned}\right\}$$

.....(30)

are the immediate result.

Any three S_1, S_2, S_3 of the four quadrics (which, subject to the condition of having a common self-reciprocal tetrahedron, may be otherwise entirely arbitrary) being supposed given, the fourth S_4 of the quartet (which is then unique and implicitly given with them) is completely determined by any of the preceding triads of equations. The elimination of any two of its four coefficients a_4, b_4, c_4, d_4 between the three equations of the triad leading (as in Art. 5° for H and K) to an equation of the fourth degree for the ratio of the remaining

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two, and three of the four resulting surfaces coinciding (as is evident from equations (26) or (27)) with the original three; thus leaving but one, the associated fourth, distinct.

When, for the three original surfaces S_1, S_2, S_3, either of the two quartets of determinants, $(b_1c_2d_3)$, $(c_1d_2a_3)$, $(d_1a_2b_3)$, $(a_1b_2c_3)$, or $\left(\frac{1}{b_1}\frac{1}{c_2}\frac{1}{d_3}\right)$, $\left(\frac{1}{c_1}\frac{1}{d_2}\frac{1}{a_3}\right)$, $\left(\frac{1}{d_1}\frac{1}{a_2}\frac{1}{b_3}\right)$, $\left(\frac{1}{a_1}\frac{1}{b_2}\frac{1}{c_3}\right)$, all vanish, the associated surface S_4, as is evident from equations (30), becomes indeterminate; as it ought, every quadric passing through their common curve of intersection in the former case, or inscribed to their common developable of circumscription in the latter case, then evidently satisfying the conditions that suffice in general to determine it.

Trinity College, Dublin,
Sept. 15, 1871.

ON GAUSS' THEOREM OF THE MEASURE OF CURVATURE AT ANY POINT OF A SURFACE.

By Benjamin Williamson, A.M.

In the following paper it is proposed to give an elementary investigation of Gauss' celebrated expression for the measure of curvature at any point on a curved surface (vide *Disquisitiones Generales circa Superficies Curvas*).

Adopting Gauss' notation, we suppose the coordinates x, y, z of any point, P, on a surface to be expressed in terms of two independent variables, u and v: thus

$$x = f(u, v),\quad y = f_1(u, v),\quad z = f_2(u, v).$$

On differentiating, we have

$$dx = adu + a'dv,\quad dy = bdu + b'dv,\quad dz = cdu + c'dv;$$

where $$a = \frac{df}{du},\quad a' = \frac{df}{dv},\quad b = \frac{df_1}{du},\ \&c.$$

Squaring and adding, we get

$$ds^2 = dx^2 + dy^2 + dz^2 = Edu^2 + 2Fdudv + Gdv^2,$$

$$E = a^2 + b^2 + c^2,\quad F = aa' + bb' + cc',\quad G = a'^2 + b'^2 + c'^2.$$

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On proceeding to second differentials, we have

$$d^2x = \alpha du^2 + 2\alpha' du\,dv + \alpha'' dv^2,$$
$$d^2y = \beta du^2 + 2\beta' du\,dv + \beta'' dv^2,$$
$$d^2z = \gamma du^2 + 2\gamma' du\,dv + \gamma'' dv^2;$$

where $$\alpha = \frac{da}{du}, \quad \alpha' = \frac{da}{dv}, \quad \&c.$$

Again, from the equation of the surface we have,

$$Ldx + Mdy + Ndz = 0;$$

on substituting their values for dx, dy, dz, and observing that u and v are independent, we get

$$La + Mb + Nc = 0, \quad La' + Mb' + Nc' = 0.$$

Hence $$L = \lambda A, \quad M = \lambda B, \quad N = \lambda C,$$

where $$A = \begin{vmatrix} b, & c \\ b', & c' \end{vmatrix}, \quad B = \begin{vmatrix} c, & a \\ c', & a' \end{vmatrix}, \quad C = \begin{vmatrix} a, & b \\ a', & b' \end{vmatrix}.$$

Now supposing any normal section drawn through the point P, it is obvious from elementary geometry that the radius of curvature (ρ) of the section is represented by $\frac{ds^2}{2p}$, where p is the infinitely small perpendicular let fall on the tangent plane at P from the extremity of the elementary arc ds.

Moreover, the equation of the tangent plane is

$$L(x' - x) + M(y' - y) + N(z' - z) = 0,$$

where x', y', z' are the current coordinates of the plane.

The length of the perpendicular from any point (X, Y, Z) on this plane is represented by

$$\frac{L(X - x) + M(Y - y) + N(Z - z)}{\sqrt{(L^2 + M^2 + N^2)}}.$$

On substituting $x + dx + \frac{d^2x}{2}$ for X, and similar expressions for Y and Z, we obtain

$$p = \frac{Ld^2x + Md^2y + Nd^2z}{2\sqrt{(L^2 + M^2 + N^2)}} = \frac{Ad^2x + Bd^2y + Cd^2z}{2\sqrt{(A^2 + B^2 + C^2)}}$$
$$= \frac{Ddu^2 + 2D_1 du\,dv + D_2 dv^2}{2\sqrt{(A^2 + B^2 + C^2)}};$$

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where $D = \begin{matrix} \alpha, & \beta, & \gamma \\ a, & b, & c \\ a', & b', & c' \end{matrix}$, $D_1 = \begin{matrix} \alpha', & \beta', & \gamma' \\ a, & b, & c \\ a', & b', & c' \end{matrix}$, $D_2 = \begin{matrix} \alpha'', & \beta'', & \gamma'' \\ a, & b, & c \\ a', & b', & c' \end{matrix}$.

Also, we have

$$A^2 + B^2 + C^2 = EG - F^2,$$

and consequently,

$$\frac{\sqrt{(EG - F^2)}}{\rho} = \frac{Ddu^2 + 2D_1 du\,dv + D_2 dv^2}{Edu^2 + 2Fdu\,dv + Gdv^2}.$$

This furnishes an expression for the radius of curvature of any normal section. In order to find the principal radii of curvature, we shall investigate the maximum and minimum values of the preceding fraction.

On making $du = \xi dv$, and denoting $\sqrt{(EG - F^2)}$ by K, we have

$$\frac{K}{\rho} = \frac{D\xi^2 + 2D_1\xi + D_2}{E\xi^2 + 2F\xi + G},$$

or
$$D\xi^2 + 2D_1\xi + D_2 = \frac{K}{\rho}(E\xi^2 + 2F\xi + G).$$

Differentiating on the hypothesis that ρ is a maximum or a minimum, we obtain

$$D\xi + D_1 = \frac{K}{\rho}(E\xi + F),$$

and it follows that the principal radii of curvature are the roots of the quadratic,

$$\begin{vmatrix} \rho D - KE, & \rho D_1 - KF \\ \rho D_1 - KF, & \rho D_2 - KG \end{vmatrix} = 0.$$

Hence denoting these roots by R and R', we have

$$\frac{1}{RR'} = \frac{DD_2 - D_1^2}{(EG - F^2)^2},$$

which furnishes an expression for the *measure of curvature* at the point.

It is easy to see that the numerator in the preceding fraction can be expressed in terms of E, F, G and their differentials. For from the expressions for D, D_1, D_2 given

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before, we obtain, by the ordinary rule for the multiplication of determinants,

$$DD_2 = \begin{vmatrix} a\alpha'' + \beta\beta'' + \gamma\gamma'', & a\alpha'' + b\beta'' + c\gamma'', & a'\alpha'' + b'\beta'' + c'\gamma'' \\ a\alpha + b\beta + c\gamma, & a^2 + b^2 + c^2, & aa' + bb' + cc' \\ a'\alpha + b'\beta + c'\gamma, & aa' + bb' + cc', & a'^2 + b'^2 + c'^2 \end{vmatrix},$$

$$D_1^2 = \begin{vmatrix} \alpha'^2 + \beta'^2 + \gamma'^2, & a\alpha' + b\beta' + c\gamma', & a'\alpha' + b'\beta' + c'\gamma' \\ a\alpha' + b\beta' + c\gamma', & a^2 + b^2 + c^2, & aa' + bb' + cc' \\ a'\alpha' + b'\beta' + c'\gamma', & aa' + bb' + cc', & a'^2 + b'^2 + c'^2 \end{vmatrix}.$$

Moreover, since

$$E = a^2 + b^2 + c^2, \quad F = aa' + bb' + cc', \quad G = a'^2 + b'^2 + c'^2,$$

we obtain

$$a\alpha + b\beta + c\gamma = \frac{1}{2}\frac{dE}{du}, \quad a\alpha' + b\beta' + c\gamma' = \frac{1}{2}\frac{dE}{dv},$$

$$a'\alpha' + b'\beta' + c'\gamma' = \frac{1}{2}\frac{dG}{du}, \quad a'\alpha'' + b'\beta'' + c'\gamma'' = \frac{1}{2}\frac{dG}{dv},$$

$$a\alpha'' + b\beta'' + c\gamma'' = \frac{dF}{dv} - (a'\alpha' + b'\beta' + c'\gamma') = \frac{dF}{dv} - \frac{1}{2}\frac{dG}{du},$$

$$a'\alpha + b'\beta + c'\gamma = \frac{dF}{du} - (a\alpha' + b\beta' + c\gamma') = \frac{dF}{du} - \frac{1}{2}\frac{dE}{dv}.$$

By aid of these equations, all the terms in the preceding determinants, except the leading term in each, can be expressed in terms of E, F, G and their differentials.

Moreover, by differentiation, we have

$$a\alpha'' + \beta\beta'' + \gamma\gamma'' + a'\frac{d\alpha}{dv} + b'\frac{d\beta}{dv} + c'\frac{d\gamma}{dv} = \frac{d^2F}{du\,dv} - \frac{1}{2}\frac{d^2E}{dv^2},$$

$$\alpha'^2 + \beta'^2 + \gamma'^2 + a'\frac{d\alpha'}{du} + b'\frac{d\beta'}{du} + c'\frac{d\gamma'}{du} = \frac{1}{2}\frac{d^2G}{du^2},$$

hence, since $\frac{d\alpha}{dv} = \frac{d\alpha'}{du}$, &c., we get

$$a\alpha'' + \beta\beta'' + \gamma\gamma'' - (\alpha'^2 + \beta'^2 + \gamma'^2) = \frac{d^2F}{du\,dv} - \frac{1}{2}\frac{d^2E}{dv^2} - \frac{1}{2}\frac{d^2G}{du^2},$$

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and, since the leading terms in the determinants are multiplied by the same factor, we have obviously

$$DD_2 - D_1^2 = \begin{vmatrix} \frac{d^2F}{du\,dv} - \frac{1}{2}\frac{d^2E}{dv^2}, & \frac{dF}{dv} - \frac{1}{2}\frac{dG}{du}, & \frac{1}{2}\frac{dG}{dv} \\ \frac{1}{2}\frac{dE}{du}, & E, & F \\ \frac{dF}{du} - \frac{1}{2}\frac{dE}{dv}, & F, & G \end{vmatrix} \cdot$$

$$- \begin{vmatrix} \frac{1}{2}\frac{d^2G}{du^2}, & \frac{1}{2}\frac{dE}{dv}, & \frac{1}{2}\frac{dG}{du} \\ \frac{1}{2}\frac{dE}{dv}, & E, & F \\ \frac{1}{2}\frac{dG}{du}, & F, & G \end{vmatrix},$$

which establishes Gauss' theorem. [See Salmon's *Surfaces*, Art. 403–7].

Trinity College, Dublin,
Sept. 21, 1871.

NOTES ON THE CIRCLE WHICH CUTS THREE GIVEN CIRCLES AT GIVEN ANGLES.

By John Griffiths.

1. If r_1, r_2, r_3 be the radii, and $\delta_1, \delta_2, \delta_3$ the distances between the centres of three given circles

$$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0, \quad S_2 = \ldots = 0, \quad S_3 = \ldots = 0,$$

we know that the quartic

$$\Omega = (BC - F^2, \ldots)(S_1, S_2, S_3)^2 = 0,$$

or, say, $\quad \Omega = (a, b, c, 2f, 2g, 2h)(S_1, S_2, S_3)^2 = 0,$

whence $a = BC - F^2$, &c., and

$$A = r_1^2 \sin^2\theta_1, \quad B = r_2^2 \sin^2\theta_2, \quad C = r_3^2 \sin^2\theta_3;$$

$$2F = -(\delta_1^2 - r_2^2 - r_3^2 + 2r_2r_3 \cos\theta_2 \cos\theta_3),$$

$$2G = -(\delta_2^2 - r_3^2 - r_1^2 + 2r_3r_1 \cos\theta_3 \cos\theta_1),$$

$$2H = -(\delta_3^2 - r_1^2 - r_2^2 + 2r_1r_2 \cos\theta_1 \cos\theta_2),$$

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represents a pair of conjugate circles cutting S_1, S_2, S_3 at the angles θ_1, θ_2, θ_3, and their supplements $\pi-\theta_1$, $\pi-\theta_2$, $\pi-\theta_3$ respectively. (See a paper by the present writer, published in the *Proceedings of the London Mathematical Society*, Vol. III. No. 37).

This being so, I first propose to shew that the quartic Ω breaks up, in general, into two factors

$$\alpha S_1+\beta S_2+\gamma S_3+\sqrt{(-k\Delta)}.J,$$

and $$\alpha S_1+\beta S_2+\gamma S_3-\sqrt{(-k\Delta)}.J=$$

where Δ is the discriminant of

$$(A, B, C, 2F, 2G, 2H)(\lambda, \mu, \nu)^2,$$

or $$\Delta=ABC-AF^2-BG^2-CH^2+2FGH,$$

$$\alpha=a'r_1\cos\theta_1+h'r_2\cos\theta_2+g'r_3\cos\theta_3,$$

$$\beta=h'r_1\cos\theta_1+b'r_2\cos\theta_2+f'r_3\cos\theta_3,$$

$$\gamma=g'r_1\cos\theta_1+f'r_2\cos\theta_2+c'r_3\cos\theta_3;$$

$$a'=B'C'-F'^2, \text{ &c., } f'=G'H'-A'F', \text{ &c.};$$

$$A'=r_1^2,\quad B'=r_2^2,\quad C'=r_3^2;$$

$$2F'=-(\delta_1^2-r_2^2-r_3^2),$$

$$2G'=-(\delta_2^2-r_3^2-r_1^2),$$

$$2H'=-(\delta_3^2-r_1^2-r_2^2),$$

and $(a', b', c', 2f', 2g', 2h')(S_1, S_2, S_3)^2=kJ^2=0$ is the equation of the Jacobian $J=x^2+y^2+\ldots$ of the given circles.

The proof of this depends mainly upon the following facts (see the paper above referred to), viz.:

1°. There is a quadric relation between S_1, S_2, S_3 and J of the form

$$kJ^2=(a', b', c', 2f', 2g', 2h')(S_1, S_2, S_3)^2.$$

2°. If the centre of the Jacobian be taken as origin of coordinates, we may write

$$J=x^2+y^2-\rho^2=0,$$

$$S_1=x^2+y^2+2g_1x+2f_1y+\rho^2=0,$$

$$S_2=x^2+y^2+2g_2x+2f_2y+\rho^2=0,$$

$$S_3=x^2+y^2+2g_3x+2f_3y+\rho^2=0,$$

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for the equations of the several circles J, S_1, S_2, S_3; or

$$S_1 = J + 2x', \quad S_2 = J + 2y', \quad S_3 = J + 2z',$$

where $x' = g_1x - fy - \rho^2$, $y' = g_2x - f_2y - \rho^2$, $z' = g_3x + f_3y + \rho^2$.

3°. The equation of J in terms of the foregoing x', y', z' is

$$(a', b', c', 2f'', 2g', 2h'')(x', y', z')^2 = 0,$$

where $a' = BC - F^2$, and $A = r_1^2$, ... $2F' = -(\delta_1^2 - r_2^2 - r_3^2)$, as before.

4°. Ω and J have a pair of two-fold intersections, which are the additional nodes of Ω.

For, excluding the circular points at infinity, the intersections Ω and J are those of the conics

$$S = (a, b, c, 2f, 2g, 2h)(x, y, z)^2 = 0,$$

$$S' = (a', b', c', 2f'', 2g', 2h')(x, y, z)^2 = 0,$$

and we have to shew that these have double contact.

Now it is evident the *reciprocal* conics Σ and Σ' have double contact, since

$$\Sigma' - \Sigma = (r_1 \cos\theta_1 \lambda + r_2 \cos\theta_2 \mu + r_3 \cos\theta_3 \nu)^2.$$

Hence (see Dr. Salmon's *Conic Sections*, p. 341, 4th edition) the equation of the conic S can be written in the form

$$(\Theta' - \Delta')\, S' - \Pi^2 = 0,$$

where $\Pi = (a'r_1 \cos\theta_1 + h'r_2 \cos\theta_2 + g'r_3 \cos\theta_3)\, x + \&c.,$

$$\Theta' = (a', b', c', 2f'', 2g', 2h')(r_1 \cos\theta_1, r_2 \cos\theta_2, r_3 \cos\theta_3)^2,$$

$$\Delta' = A'B'C' - A'F'^2 - B'G'^2 - C'H'^2 + 2F'G'H'.$$

It follows, then, in an analogous manner, that $\Omega = 0$ may be written

$$\Omega = (\Theta' - \Delta')\, kJ^2 - (\alpha S_1 + \beta S_2 + \gamma S_3)^2 = 0;$$

where $\Delta' - \Theta' = \Delta$, Δ being the discriminant of

$$(A, B, C, \ldots)(\lambda, \mu, \nu)^2 \text{ and } A = r_1^2 \sin^2\theta_1, \text{ \&c.,}$$

$$2F' = -(\delta_1^2 - r_2^2 - r_3^2 + 2r_2 - r_3 \cos\theta_2 \cos\theta_3), \text{ as before;}$$

$$\alpha = a'r_1 \cos\theta_1 + h'r_2 \cos\theta_2 + g'r_3 \cos\theta_3,$$

$$\beta = h'r_1 \cos\theta_1 + b'r_2 \cos\theta_2 + f'r_3 \cos\theta_3,$$

$$\gamma = g'r_1 \cos\theta_1 + f'r_2 \cos\theta_2 + c'r_3 \cos\theta_3,$$

and $$(a', b', \ldots)(S_1, S_2, S_3)^2 = kJ^2;$$

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that is Ω breaks up, in general, into the two factors

$$\alpha S_1 + \beta S_2 + \gamma S_3 + \delta J,\ \alpha S_1 + \beta S_2 + \gamma S_3 - \delta J;$$

where α, β, γ have the values given above, and $\delta = \sqrt{(-k\Delta)}$.

(N.B. This result will not be necessarily true when the given circles S_1, S_2, S_3 meet in a point, since, in that case, the quantities a', b', c', &c. become indeterminate).

Ex. If S_1, S_2, S_3 be taken to be the circles described upon the sides of a triangle ABC as diameters, we may write for their respective equations in trilinear coordinates

$$S_1 = \sin A\,(x^2 \cos A - yz - zx \cos C - xy \cos B) = 0,$$

$$S_2 = \sin B\,(y^2 \cos B - yz \cos C - zx - xy \cos A) = 0,$$

$$S_3 = \sin C\,(z^2 \cos C - yz \cos B - zx \cos A - xy) = 0,$$

and for that of the Jacobian

$$J = x^2 \sin A \cos A + y^2 \sin B \cos B + z^2 \sin C \cos C = 0,$$

where A, B, C are the angles of the triangle.

$$\text{Also } \left.\begin{aligned} A' &= R^2 \sin^2 A \\ B' &= R^2 \sin^2 B \\ C' &= R^2 \sin^2 C \end{aligned}\right\},\quad \left.\begin{aligned} F' &= \sin B \sin C \cos A \\ G' &= \sin C \sin A \cos B \\ H' &= \sin A \sin B \cos C \end{aligned}\right\},$$

where R = radius of circle (ABC). Hence

$$a' = R^4 \sin^2 A \sin^2 B \sin^2 C = b' = c';$$

$$f' = R^4 \sin^2 A \sin B \sin C\,(\cos B \cos C - \cos A),$$

with similar values for g' and h'. And if

$$\theta_1 = \frac{\pi}{2} - A,\ \theta_2 = \frac{\pi}{2} - B,\ \theta_3 = \frac{\pi}{2} - C,$$

$$\Theta' = R^6 \sin^2 A \sin^2 B \sin^2 C \times \{\Sigma \sin^4 A + 2\Sigma \sin B \sin C (\cos B \cos C - \cos A)\},$$

the two expressions involving Σ being written, for shortness, for

$$\sin^4 A + \sin^4 B + \sin^4 C, \text{ &c.}$$

$$\Delta' = R^6 \sin^2 A \sin^2 B \sin^2 C \times (1 + 2 \cos A \cos B \cos C - \cos^2 A - \cos^2 B - \cos^2 C),$$

and therefore

$$\Theta' - \Delta' = 4R^6 \cos^2 A \cos^2 B \cos^2 C \sin^2 A \sin^2 B \sin^2 C.$$

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provided $\lambda r_1 \cos\theta_1 + \mu r_2 \cos\theta_2 + \nu r_3 \cos\theta_3 = 0$; that is, provided the circle in question passes through the pair of points

$$\frac{S_1}{r_1 \cos\theta_1} = \frac{S_2}{r_2 \cos\theta_2} = \frac{S_3}{r_3 \cos\theta_3};$$

which must, therefore, be the *limiting* points of the system Ω.

Hence we have the following proposition:

The limiting points of the four pairs of circles which can be drawn to intersect three given circles $S_1 = x^2 + y^2 + \ldots = 0$, $S_2 = 0$, $S_3 = 0$, at the three given angles coincide with the four pairs of inverse points

$$\frac{S_1}{r_1 \cos\theta_1} = \frac{S_2}{\pm r_2 \cos\theta_2} = \frac{S_3}{\pm r_3 \cos\theta_3}.$$

It follows from this, that these points are the intersections of an infinite number of bicircular quartics represented by the equation $lS_1^2 + mS_2^2 + nS_3^2 = 0$, where l, m, n satisfy the relation

$$lr_1^2 \cos^2\theta_1 + mr_2^2 \cos^2\theta_2 + nr_3^2 \cos^2\theta_3 = 0.$$

In particular, the limiting points in question all lie on a circular cubic passing through the radical centre of the given circles S_1, S_2, S_3, viz. the circular cubic

$$\Sigma (r_2^2 \cos^2\theta_2 - r_3^2 \cos^2\theta_3) S_1^2 = 0;$$

for this evidently represents a cubic, since the coefficients of the highest powers of x and y vanish; and the equation is satisfied by $S_1 = S_2 = S_3$.

3. If U and V represent a pair of conjugate circles cutting S_1, S_2, S_3 at given angles θ_1, θ_2, θ_3, the envelope of a circle $\lambda S_1 + \mu S_2 + \nu S_3$ cutting U and V at an angle θ_4 is another pair of circles coaxal with U and V.

For, if we write for the equations of the circles U, S_1, S_2, S_3,

$$U = x^2 + y^2 - R^2,$$

$$S_1 = x^2 + y^2 + 2g_1 x + 2f_1 y + R^2 - 2Rr_1 \cos\theta_1 = 0,$$

$$S_2 = x^2 + y^2 + 2g_2 x + 2f_2 y + R^2 - 2Rr_2 \cos\theta_2 = 0,$$

$$R_3 = x^2 + y^2 + 2g_3 x + 2f_3 y + R^2 - 2Rr_3 \cos\theta_3 = 0,$$

it may be shewn that the circle $\lambda S_1 + \mu S_2 + \nu S_3$ will intersect U at an angle θ_4, provided

$$A'\lambda^2 + B'\mu^2 + C'\nu^2 + 2F'\mu\nu + 2G'\nu\lambda + 2H'\lambda\mu = (r_1 \cos\theta_1 \lambda + r_2 \cos\theta_2 \mu + r_3 \cos\theta_3 \nu)^2 \sec^2\theta_4;$$

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[where, as before, $A' = r_1^2$, ... $2F' = -(\delta_1^2 - r_2^2 - r_3^2)$...], or provided, say, that

$$A''\lambda^2 + B''\mu^2 + C''\nu^2 + 2F''\mu\nu + 2G''\nu\lambda + 2H''\lambda\mu = 0.$$

Similarly, the circle in question will intersect V at the same angle when λ, μ, ν satisfy the relation just written.

The envelope, then, of the circle $\lambda S_1 + \mu S_2 + \nu S_3$ subject to this condition is given by

$$\Omega'' = (B''C'' - F''^2, \ldots)(S_1, S_2, S_3)^2 = 0;$$

which, evidently, represents a pair of circles coaxal with the pair $\Omega = U.V$, since Ω meets the Jacobian $J(S_1, S_2, S_3)$ in the same two-fold points as Ω.

4. It may be observed that if $\theta_1 = \theta_2 = \theta_3$, then the limiting points of the system U, V are given by the equations

$$\frac{S_1}{r_1} = \frac{S_2}{r_2} = \frac{S_3}{r_3};$$

in which case their position is independent of the equal angle at which U and V intersect the given circles S_1, S_2, S_3.

If the limiting points of U, V lie on the Jacobian $J(S_1, S_2, S_3)$, it is evident that they become coincident; or, in other words, that U and V *touch* each other.

Hence the condition that the circles U and V shall touch, is

$$(a', b', c', 2f', 2g', 2h')(r_1 \cos\theta_1, r_2 \cos\theta_2, r_3 \cos\theta_3)^2 = 0,$$

where $a' = B'C' - F'^2$, $b' = \ldots$ &c.,

and $A' = r_1^2$, $B' = r_2^2$, $C' = \ldots$,

$2F' = -(\delta_1^2 - r_2^2 - r_3^2)$, $2G' = \ldots$, $2H' = \ldots$, as before.

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ON A PAIR OF DEFINITE INTEGRALS.

By WILLIAM WALTON, M.A., Fellow of Trinity Hall.

AN investigation of the values of the definite integrals

$$u=\int_0^\infty e^{-ax}x^{n-1}\cos bx\,dx,\quad v=\int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx,$$

where b is an indeterminate constant, and a, n, any real positive constants, has been inserted in the ninth volume of the *Journal de l'Ecole Polytechnique*, by Poisson, who says, "Ces formules sont dues à Euler,* qui les a trouvées par une sorte d'induction fondée sur le passage des quantités réelles aux imaginaires; induction qu' on peut bien employer comme un moyen de découverte, mais dont les résultats ont besoin d'être confirmés par des méthodes directes et rigoureuses."

The following investigation is similar to Poisson's, but more simple, inasmuch as in his analysis he has recourse to differential equations of the second order, while mine involves equations of only the first order. Although the objection to the evaluation of these and other integrals by passing from real to imaginary quantities, has, I suppose, been obviated by the researches of Cauchy,† still the following modification of Poisson's evaluation may have at least, in virtue of his name, some historical interest.

We have $$\frac{du}{db}=-\int_0^\infty e^{-ax}x^n\sin bx\,dx;$$

but $$\int e^{-ax}\sin bx\,dx=-\frac{e^{-ax}}{a^2+b^2}\cdot(a\sin bx+b\cos bx);$$

hence, integrating by parts, we see that, between the limits,

$$\frac{du}{db}=\frac{e^{-ax}x^n}{a^2+b^2}(a\sin bx+b\cos bx)-\frac{n}{a^2+b^2}\int e^{-ax}x^{n-1}(a\sin bx+b\cos bx)dx$$

$$=-\frac{n}{a^2+b^2}\int_0^\infty e^{-ax}x^{n-1}(a\sin bx+b\cos bx)\,dx$$

$$=-\frac{n}{a^2+b^2}(av+bu)\qquad\qquad(1).$$

* Tome IV. *de son Calcul. Intégral*, p. 337, et suivantes.

† See Moigno, *Calcul Différentiel et Calcul. Intégral*, Tome II. p. 302.

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Again $$\frac{dv}{db} = \int_0^\infty e^{-ax} x^n \cos bx\, dx :$$

but $$\int e^{-ax} \cos bx\, dx = \frac{e^{-ax}}{a^2 + b^2} (b \sin bx - a \cos bx) ;$$

hence, integrating by parts, we have, between the limits,

$$\frac{dv}{db} = \frac{e^{-ax} x^n}{a^2 + b^2} (b \sin bx - a \cos bx)$$

$$- \frac{n}{a^2 + b^2} \int e^{-ax} x^{n-1} (b \sin bx - a \cos bx)\, dx$$

$$= - \frac{n}{a^2 + b^2} \int_0^\infty e^{-ax} x^{n-1} (b \sin bx - a \cos bx)\, dx$$

$$= - \frac{n}{a^2 + b^2} (bv - au) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From the equations (1) and (2), we have

$$u \frac{du}{db} + v \frac{dv}{db} = - nb . \frac{u^2 + v^2}{a^2 + b^2} ,$$

whence $$u^2 + v^2 = \frac{C}{(a^2 + b^2)^n} ,$$

where C is independent of b: but, when $b = 0$, it is evident that, the value of u being $\Gamma(n)$, that of v is zero. Hence

$$u^2 + v^2 = \frac{\{\Gamma(n)\}^2}{(a^2 + b^2)^n} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Again, from (1) and (2), we see that

$$u \frac{dv}{db} - v \frac{du}{db} = na \frac{u^2 + v^2}{a^2 + b^2} ,$$

$$\frac{\frac{d}{db}\left(\frac{v}{u}\right)}{1 + \left(\frac{v}{u}\right)^2} = \frac{na}{a^2 + b^2} ,$$

$$\tan^{-1} \frac{v}{u} = C + n \tan^{-1} \frac{b}{a} ,$$

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and therefore, since $\frac{v}{u} = 0$ when $b = 0$,

$$\frac{v}{u} = \tan\left(n \tan^{-1}\frac{b}{a}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

The equations (3) and (4) shew that

$$u = \frac{\Gamma(n) \cos\left(n \tan^{-1}\frac{b}{a}\right)}{(a^2+b^2)^{\frac{1}{2}n}}, \quad v = \frac{\Gamma(n) \sin\left(n \tan^{-1}\frac{b}{a}\right)}{(a^2+b^2)^{\frac{1}{2}n}}.$$

July 20, 1871.

REVIEW.

Tables de Logarithmes vulgaires à dix decimales construites d'après un nouveau mode par S. PINETO, approuvées par l'Academie des Sciences de S. Petersbourg. S. Petersbourg, 1871.

THE tables occupy 56 pages—the principal one being a table in 44 pages, the 22 left-hand pages containing the 10 figure logarithms of the numbers from 1,000,000 to 1,010,999, and the 22 right-hand pages the proportional parts ·01,·02,...·99 of the differences. A like table 100,000 to 999,999 would occupy 3600 pages. By means of an auxiliary table of 3 pages, and of a slight increase of the numerical calculation, *the table of* 44 *pages does the work of the table of* 3600 *pages*. To explain how this is: the auxiliary table gives for any number A the initial four digits of which are equal to or exceed 1011, a multiplier M, such that in the product MA the initial four digits are between 1000 and 1011; this multiplier M contains only 1, 2 or 3 figures, and when there are 3 figures, then in general either the middle figure is 0, or two of the figures are equal; the table gives also $\log\frac{1}{M}$ to 12 decimals; and there is a third column as will be explained. Hence A being as above, the auxiliary table gives M, we form the product MA, obtain the logarithm thereof from the principal table, and adding thereto $\log\frac{1}{M}$, we have the required $\log A$. Conversely, when there is

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given a logarithm B the first five digits in the mantissa of which are not included between 00000 and 00474 (being the limits of the first five digits of the logarithms in the principal table), the auxiliary table by means of its third column gives M; adding $\log \frac{1}{M}$ to B, we have a logarithm included in the limits of the principal table, and seeking for the corresponding number, this is $= \frac{1}{M}$ number having B for its logarithm: that is, the required number is $= M$ times the number obtained as above. Of course as regards the principal Table, the proportional parts are employed in the usual manner; the tabulation of them to hundredths (instead of tenths) facilitates the interpolation; for better securing the accuracy of the last figure, directions are given in regard to the 11th and 12th figures. An example of the determination of a logarithm is as follows:

$$\pi = \quad 3{\cdot}14159\ 26536$$
$$M = 32$$
$$M\pi = \ 100.53096\ 49152$$

$$\log \frac{1}{M} = 8{\cdot}49485\ 00216.80 - 10$$

log =	2·00229 95705.75	$D = 4320$
64	2764.80	
91	39.31	
52	22	
$\log \pi =$	0·49714 98726.88	

(correct value of last two figures = 94).

The labour saved by the small bulk of the Tables goes far to balance that occasioned by the additional steps in the calculation.

THE END OF VOL. XI.

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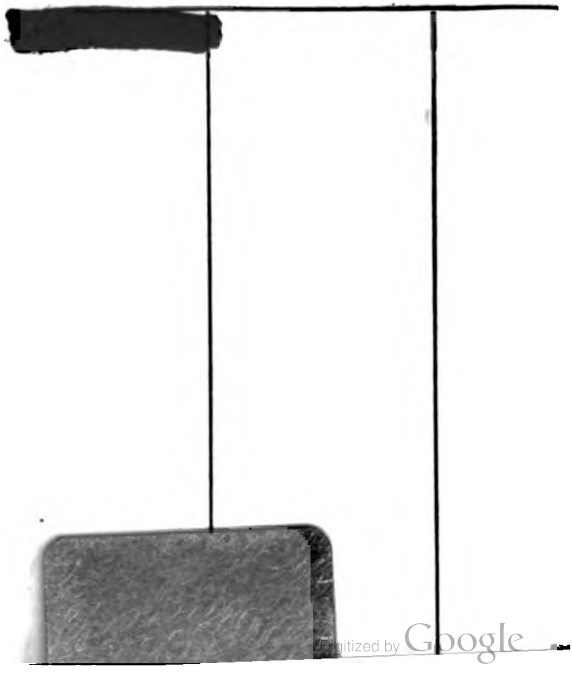

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